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REDUCTION EXPONENT AND DEGREE BOUND FOR THE DEFINING EQUATIONS OF GRADED RINGS

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ABSTRACT. The paper gives upper degree bounds for the defining equations of certain graded rings in terms of the reduction exponent and the multiplicity.

1. Introduction. Let $(A, \mathbf{m})$ be a local ring and $\mathbf{a}$ an $\mathbf{m}$-primary ideal. Let $G_{\mathbf{a}}(A)$ denote the associated graded ring $\bigoplus_{n=0}^{\infty} \mathbf{a}^n/\mathbf{a}^{n+1}$. Hong [**7**], Cho [**2**], and Achilles and Schenzel [**1**] have shown that if A is a one-dimensional Buchsbaum (resp. Cohen-Macaulay) ring, then the degrees of the defining equations of $G_{\mathbf{a}}(A)$ represented as a quotient of a polynomial ring over $A/\mathbf{a}$ are bounded above by $e(A)+1$ (resp. $e(A)$), where $e(A)$ denotes the multiplicity of A with respect to $\mathbf{m}$. The main goal of this paper is to generalize this result for higher-dimensional Buchsbaum (resp. Cohen-Macaulay) rings. Moreover, we also want to explain the phenomenon that all the above works employed, in nonapparent and different ways, the reduction exponent of a minimal reduction of $\mathbf{a}$ and obtained its invariance as a by-product. This is of some interest because Sally [**13**] has discussed the problem whether A being Cohen-Macaulay implies the invariance of the reduction exponent of minimal reductions of $\mathbf{m}$. Recall that an ideal $\mathbf{b} \subseteq \mathbf{a}$ is called a reduction of $\mathbf{a}$ if $\mathbf{a}^{n+1} = \mathbf{b}\mathbf{a}^n$ for some nonnegative integer n [**9**] and that the reduction exponent of $\mathbf{b}$ is the least integer n with this property [**11, 12**]. (For the theory of Buchsbaum rings see [**14**].)

To achieve the above goal we will study the relationship between the reduction exponent $r(I)$ of a minimal reduction I of the positively graded part of a graded ring $R = \bigoplus_{n=0}^{\infty} R_n$ and the degrees of the defining equations of R represented as a quotient of a polynomial ring over R_0. It will turn out that these degrees are bounded above by $\max\{r(I)+1, a(I)\}$, where $a(I)$ denotes the least integer n such that I can be generated by a sequence of elements which is regular in degree $\geq n$. This bound is independent of the choice of I because it can be expressed by means of the local cohomology modules of R in a manner like Castelnuovo's regularity (recently studied by Ooishi [**10**], and Eisenbud and Goto [**4**]). If I can be generated by a d-sequence [**8**], we always have $r(I)+1 \geq a(I)$. This implies the invariance of $r(I)$ and the degree bound $r(I)+1$ for the defining equations of a graded Buchsbaum ring R with R_0 being a zero-dimensional local ring. In particular, by comparing $r(I)$ with the multiplicity $e(R)$ of R with respect to the maximal graded ideal, we shall obtain the following result.

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THEOREM 1.1. *Let R be a graded Buchsbaum (resp. Cohen-Macauley) ring with R_0 a zero-dimensional local ring. Then the degrees of the defining equations of R are bounded above by $e(R)+1$ (resp. $e(R)$).*

A modified version of the above method applied to $G_{\mathbf{a}}(A)$ will yield the following generalization of the results of Hong, Cho, Achilles, and Schenzel.

THEOREM 1.2. *Let A be a Buchsbaum* (resp. Cohen-Macaulay) ring with $d = \dim(A) \geq 1$ and $\operatorname{depth}(G_{\mathbf{a}}(A)) \geq d-1$. Let $r(\mathbf{b})$ denote the reduction exponent of a minimal reduction $\mathbf{b}$ of $\mathbf{a}$. Then

(i) $r(\mathbf{b})$ is independent of the choice of $\mathbf{b}$,

(ii) the degrees of the defining equations of $G_{\mathbf{a}}(A)$ are bounded above by $r(\mathbf{b})+1$,

(iii) $r(\mathbf{b}) \leq e(A)t^{d-1}$ (resp. $e(A)t^{d-1} - 1$), where t is the least integer with the property $\mathbf{m}^t \subseteq \mathbf{a}$.

We would like to mention that a special situation of the Cohen-Macaulay case of Theorem 1.2 was already considered by Cho [**2**] who claimed that if moreover $G_{\mathbf{a}}(A)$ is free over $A/\mathbf{a}$, the degrees of the defining equation of $G_{\mathbf{a}}(A)$ are bounded above by $e(A)$. But the proof for that contains an error which could not be corrected if $d > 1$ (see Remark 5.2).

The proofs of the above theorems will be found in §§4 and 5, respectively. §§2 and 3 only have a preparatory character.

2. Filter-regular sequence. Throughout this paper, let $R = \bigoplus_{n=0}^{\infty} R_n$ be a noetherian commutative graded ring such that R_0 is a lock ring and the R_0-algebra R is generated by the elements of R_1. Let M denote the unique maximal graded ideal of R and k the residue field R/M. Put $R^+ = \bigoplus_{n\geq 1} R_n$. If E is a graded module over R, then we denote by E_n the nth graded piece of E.

Let $\mathbf{f} = f_1, \ldots, f_r$ be a sequence of homogeneous elements of R, $r \geq 1$. One can use the following notion to measure how much $\mathbf{f}$ differs from being a regular sequence.

DEFINITION. $\mathbf{f}$ is called n-regular if

$$((f_1, \ldots, f_{i-1})\colon f_i)_n = (f_1, \ldots, f_{i-1})_n$$

for $i = 1, \ldots, r$. The least integer m such that $\mathbf{f}$ is n-regular for all $n \geq m$ will be denoted by $a(\mathbf{f})$.

We shall see that the condition $a(\mathbf{f}) < \infty$ can be characterized by the behavior of f_i towards the associated primes of $(f_1, \ldots, f_{i-1})$, $i = 1, \ldots, r$.

DEFINITION. $\mathbf{f}$ is called filter-regular (with respect ro R^+) if $f_i \notin P$ for all primes $P \in \operatorname{Ass}(R/(f_1, \ldots, f_{i-1}))$, $P \not\supseteq R^+$, $i = 1, \ldots, r$.

This notion has its origin in the theory of Buchsbaum rings [**3**] and is closely related to the notion of d-sequences [**15**].

LEMMA 2.1. *$a(\mathbf{f}) < \infty$ if and only if $\mathbf{f}$ is filter-regular.*

PROOF. $((f_1, \ldots, f_{i-1})\colon f_i)_n = (f_1, \ldots, f_{i-1})_n$ for n sufficiently large if there exists an integer m such that

$$(R^+)^m((f_1, \ldots, f_{i-1})\colon f_i) \subseteq (f_1 \ldots, f_{i-1})$$

or, equivalently,

$$(f_1, \dots, f_{i-1})\colon f_i \subseteq \bigcup_{n=0}^{\infty} (f_1, \dots, f_{i-1})\colon (R^+)^n.$$

Since $\bigcup_{n=0}^{\infty}(f_1, \dots, f_{i-1})\colon (R^+)^n$ is the intersection of all primary components of $(f_1, \dots, f_{i-1})$ whose associated primes do not contain R^+, the last inclusion is satisfied iff $f_i \notin P$ for all primes $P \in \operatorname{Ass}(R/(f_1, \dots, f_{i-1}))$, $P \not\supseteq R^+$.

If $a(\mathbf{f}) < \infty$ and $f_1, \dots, f_r \in R_1$, $a(\mathbf{f})$ can only take a certain value depending on the number r. To see this define

$$a_i(R) = \inf\{m \in Z; H^i_{R^+}(R)_n = 0 \text{ for all } n \geq m\}$$

for all $i \geq 0$, where $H^i_{R^+}(R)$ denotes the ith local cohomology module of R with respect to R^+. For short, we will set $a_i = a_i(R)$.

PROPOSITION 2.2. *Let $\mathbf{f}$ be a filter-regular sequence in R_1. Then*

$$a(\mathbf{f}) = \max\{a_i + i\colon i = 0, \dots, r-1\}.$$

We shall need the following lemma in the proof of Proposition 2.2.

LEMMA 2.3. *Let G be a filter-regular element in R_1. Then*

$$a_{i+1} + 1 \leq a_i(R/gR) \leq \max\{a_1, a_{i+1} + 1\}$$

for all $i \geq 0$.

PROOF. By the definition of a filter-regular sequence, $0 : g \subseteq \bigcup_{n=0}^{\infty} 0 : (R^+)^n$. Therefore, $0 : g$ is annihilated by some power of R^+. Hence $H^i_{R^+}(0 : g) = 0$ for $i \geq 1$. Hence from the exact sequence

$$0 \longrightarrow 0 : g \longrightarrow R \longrightarrow R/0 : g \longrightarrow 0$$

we get $H^i_{R^+}(R) = H^i_{R^+}(R/0 : g)$ for $i \geq 1$. Now, from the exact sequence

$$0 \longrightarrow R/0 : g \xrightarrow{g} R \longrightarrow R/gR \longrightarrow 0$$

one can derive the exact sequence

$$H^i_{R^+}(R)_n \rightarrow H^i_{R^+}(R/gR)_n \rightarrow H^{i+1}_{R^+}(R)_{n-1} \rightarrow H^{i+1}_{R^+}(R)_n$$

for $i \geq 0$ which immediately implies the statement.

PROOF OF PROPOSITION 2.2. Since

$$0 : f_1 \subseteq \bigcup_{n=0}^{\infty} 0 : (R^+)^n \quad \text{and} \quad f_1 H^0_{R^+}(R)_{a_0 - 1} \subseteq H^0_{R^+}(R)_{a_0} = 0,$$

$a(f_1) = a_0$. Therefore, the case $r = 1$ is immediate. For $r > 1$ let $\mathbf{f}'$ denote the sequence of the images of $f_2, \dots, f_r$ in R/f_1R. By induction and using Lemma 2.3, we have

$$\begin{aligned}\max\{a_i + i; i = 1, \dots, r-1\} \leq a(\mathbf{f}') &= \max\{a_i(R/f_1R) + i; i = 0, \dots, r-2\} \\ &\leq \max\{a_i + i; i = 0, \dots, r-1\}.\end{aligned}$$

Thus, since $a(\mathbf{f}) = \max\{a(f_1), a(\mathbf{f}')\}$, the statement is obvious.

3. Reduction exponent. Let $\mathbf{n}$ be the maximal ideal of R_0. By tensoring R with $R_0[u]_{(n)}$, where u is some indeterminate, we may always assume that k is infinite. Then there exist minimal reductions of R^+ [**9**]. Let I be an arbitrary minimal reduction of R^+. The reduction exponent $r(I)$ is just the least integer n such that $R_{n+1} = I_{n+1}$. The aim of this section is to produce some relationship between $r(I)$ and the local cohomology modules of R.

Set $r = \dim(R/\mathbf{n}R)$, the analytic spread of R^+. It is well known that every homogeneous minimal basis of I consists of r linear forms.

LEMMA 3.1. *There exist filter regular sequences of R minimally generating I.*

PROOF. This statement may be formulated for every homogeneous ideal J with the property $\sqrt{J} = \sqrt{R^+}$. Indeed, by induction on $\dim_k(J/MJ)$, we only need to choose a homogeneous element $g \in J\backslash MJ$ such that $g \notin P$ for all primes $P \in \operatorname{Ass}(R)$, $P \not\supseteq R^+$. That is always possible because k is infinite and $J\backslash MJ$ is not contained in any P by the Nakayama lemma.

PROPOSITION 3.2. $a_r + r \leq r(I) + 1 \leq \max\{a_i + i; i = 0, \ldots, r\}$.

PROOF. If $r = 0$, R^+ is nilpotent and $r(I)$ is the least integer n such that $R_{n+1} = 0$. Therefore, $r(I) + 1 = a_0$ because $H^0_{R^+}(R) = R$. If $r > 1$, choose a filter-regular sequence $f_1, \ldots, f_r$ generating I. Note that I/f_1R is a minimal reduction of R^+/f_1R with $r(I/f_1R) = r(I)$. Then the statement will follow from the induction hypothesis on R/f_1R by using Lemma 2.3.

At this point we remark that $\max\{a_i + i; i = 0, \ldots, r\} - 1$ is the Castelnuovo regularity of R [**10**]. In particular, if we define

$$a(I) := \inf\{a(\mathbf{f}); I = (\mathbf{f})\},$$

then we have the following relation.

COROLLARY 3.3. $\max\{r(I) + 1, a(I)\} = \max\{a_i + i; i = 0, \ldots, r\}$.

PROOF. By Lemma 2.1, Proposition 2.2, and Lemma 3.1,

$$a(I) = \max\{a_i + i; i = 0, \ldots, r - 1\}.$$

Hence the statement follows from Proposition 3.2.

LEMMA 3.4. *Suppose that I is minimally generated by a d-sequence* $\mathbf{f}$. *Then* $r(I) + 1 \geq a(I)$.

PROOF. By the definition of a d-sequence,

$$((f_1, \ldots, f_{i-1})\colon f_i) \cap I = (f_1, \ldots, f_{i-1})$$

for $i = 1, \ldots, r$ [**8**, **15**]. If $n \geq r(I) + 1$, then $R_n = I_n$ and hence

$$((f_1, \ldots, f_{i-1})\colon f_i)_n = ((f_1, \ldots, f_{i-1})\colon f_i)_n \cap I_n = (f_1, \ldots, f_{i-1})_n.$$

In the following we call R a graded Buchsbaum ring if R_M is a Buchsbaum local ring [**14**].

COROLLARY 3.5. *Suppose that R is a graded Buchsbaum ring with* $\dim(R_0) = 0$. *Then* $r(I) + 1 = \max\{a_i + i; i = 0, \dots, r\}$ *(independent of the choice of I).*

PROOF. Since $\dim(R_0) = 0$, I can be generated by a homogeneous sytem of parameters of R. By [**8**], R is Buchsbaum iff every stystem of parameters of R is a d-sequence. Hence the statement follows from Lemma 3.4.

Corollary 3.5 is a generalization of the formula $r(I) = a_d + d - 1$ for Cohen-Macaulay rings R [**5**]. In general, if $r(I)$ is independent of the choice of I, $r(I)$ need not be $\max\{a_i + i; i = 0, \dots, r\} - 1$.

EXAMPLE. Let $R = k[X, Y]/(X^2, XY^2)$. Then every minimal reduction of R^+ has the reduction exponent 1, whereas $a_0 = 3$.

4. Degree bound. Let $f_1, \dots, f_s$ be a minimal basis of the R_0-module R_1. Then one can define a natural map from the polynomial ring

$$R_0[X] := R_0[X_1, \dots, X_s]$$

to R by sending X_i to f_i, $i = 1, \dots, s$. Let P denote the kernel of this map. It is not hard to see that the set of integers occurring as degrees of elements of homogeneous minimal bases of P is independent of the choice of $f_1, \dots, f_s$. Let m_R denote the maximum of these integers.

PROPOSITION 4.1. $m_R \leq \max\{r(I) + 1, a(I)\}$ *for every minimal reduction I of R^+.*

PROOF. Let Q denote the ideal of all elements of $R_0[X]$ whose images in R belong to I. Let m denote the maximum degree of elements of all homogeneous minimal bases of Q. Since $R_0[X]_n = Q_n$ for $n \geq r(I) + 1$, $m \leq r(I) + 1$. Hence we may assume that $m_R > m$. Choose r linear forms $F_1, \dots, F_r$ such that the sequence $\mathbf{f}$ of their images in R forms a basis of I with $a(\mathbf{f}) = a(1)$. Let $F \in P$ be an arbitrary form of degree m_R. Since $F_1, \dots, F_r$ can be extended to a minimal basis of Q by forms of P, $F = G + \sum_{i=1}^r F_i G_i$ for some forms G and G_i such that G belongs to the ideal Q' generated by forms of P of degree $< m_R$. If $m_R > a(I)$,

$$G_r \in ((P, F_1, \dots, F_{r-1}) \colon F_r)_{m_R - 1} = (P, F_1, \dots, F_{r-1})_{m_R - 1}.$$

Hence we can omit the term $G_r F_r$ in the above presentation of F. Proceeding like that, we can successively omit $G_{r-1}F_{r-1}, \dots, G_1 F_1$ and obtain $F \in Q'$, a contradiction.

Combining Proposition 4.1 with Corollary 3.3, we get

$$m_R \leq \max\{a_i + i; i = 0, \dots, r\}$$

which generalizes a result of Ooishi [**10**, Proposition 20], and Eisenbud and Goto [**4**, Theorem 1.2]. In particular, by Corollary 3.5, we have the following consequence.

COROLLARY 4.2. *Let R be a graded Buchsbaum ring with* $\dim(R_0) = 0$. *Then* $m_R \leq r(I) + 1$ *for any minimal reduction I of R^+.*

PROOF OF THEOREM 1.1. By Corollary 4.2, it suffices to show that $r(I) \leq e(R)$ (resp. $e(R) - 1$). Since $\dim(R_0) = 0$, I is a minimal reduction of M. Hence $e(R)$ is equal to the multiplicity of R with respect to I. Note that I is generated by a homogeneous system of parameters $\mathbf{f} = f_1, \dots, f_r$ of R. If R is Cohen-Macaulay, $r(I) \leq l(R/I) - 1 = e(R) - 1$. If R is Buchsbaum and $r = 1$, consider the ring

$R' = R/H^0_m(R)$. Then $r(IR') \leq e(R') - 1$ because R' is a Cohen-Macaulay ring. Since $R^+H^0_M(R) = 0$ [**14**], it is easy to to check that $r(IR') \geq r(I) - 1$ and $e(R') = e(R)$. Hence $r(I) \leq e(R)$. If $r > 1$, consider the ring R/f_1R. Then $r(I/f_1R) = r(I)$ and by the definition of Buchsbaum rings, $e(R/f_1R) = e(R)$. Hence $r(I) \leq e(R)$ by the induction hypothesis of R/f_1R.

It is worth noticing that the bound $r(I)+1$ of Corollary 4.2 is much better than the bound $e(R)+1$ of Theorem 1.1.

EXAMPLE. Let R be the homogeneous coordinate ring of a projective curve given parametrically by a set S of monomials of degree e in two indeterminates t, s such that $r^e, t^{e-1}s, ts^{e-1}, s^e \in S$. Consider R as the subring of $k[t,s]$ generated by the monomials of S. Put $I = (t^e, s^e)$. Then $r(I) \leq 2$ if R is a Buchsbaum ring [**16**, Theorem 4.1], whereas $e(R) = e$ may be arbitrarily large. A famous example is the twisted cubic curve with the parameters t^4, t^3s, ts^3, s^4 which is known to be arithmetically Buchsbaum.

If $R_0 = k$, Ooishi [**10**, Theorems 19 and 20] has given the bound $m_R \leq e(R) + r - \dim(R_1) + i(R) + 1$, where $i(R) := \sum_{i=0}^{r-1} \binom{r-1}{i} H^i_M(R)$. Except in the case where R is a Cohen-Macaulay ring, this bound is not better than the bound $e(R)+1$ of Theorem 1.1.

EXAMPLE. Let R be as in the above example with $S = \{t^{3p-i+1}s^i; i = 0, 1, 3p, 3p+1 \text{ and } i = 3j-1 \text{ with } j = 1, \dots, p\}$, where p is some positive integer. By [**16**, Theorem 4.1], it is easy to check that R is a Buchsbaum ring with $i(R) = 2p-2$. Therefore $e(R) + 1 = 3p + 2 < 4p - 2 = e(R) + r - \dim(R_1) + i(R) + 1$ if $p > 4$.

5. The local case. Let $\mathbf{a}$ be an arbitrary ideal of the local ring $(A, \mathbf{m})$. Set $r = \dim(G_{\mathbf{a}}(A)/\mathbf{m}G_{\mathbf{a}}(A))$, the analytic spread of $\mathbf{a}$. Then $\mathbf{a}$ is called equimultiple if $\operatorname{ht}(\mathbf{a}) = r$ [**6**]. This notion has some relation to Zariski's equimultiplicity and is satisfied if A is normal flat along $\mathbf{a}$ (an important notion in Hironaka's resolution of singularities). The class of equimultiple ideals is rather large. It contains e.g. all $\mathbf{m}$-primary ideals.

PROPOSITION 5.1. *Let A be a Buchsbaum ring and $\mathbf{a}$ an equimultiple ideal. Let $x_1, \dots, x_r$ be elements of $\mathbf{a}$ such that $\mathbf{b} = (x_1, \dots, x_r)$ is a minimal reduction of $\mathbf{a}$ and*

$$(x_1, \dots, x_i) \cap \mathbf{a}^{n+1} = (x+1, \dots, x_i)\mathbf{a}^n$$

for all $n \geq 0$, $i = 1, \dots, r-1$. Let $r(\mathbf{b})$ denote the reduction exponent of $\mathbf{b}$. Set $R = G_{\mathbf{a}}(A)$. Then

(i) $r(\mathbf{b}) = \max\{a_i + i; i = 0, \dots, r\} - 1$,

(ii) $m_R \leq r(\mathbf{b}) + 1$.

PROOF. Let $f_1, \dots, f_r$ be the initial forms of $x_1, \dots, x_r$ in R. Then $I = (f_1, \dots, f_r)$ is a minimal reduction of R^+ with $r(I) = r(\mathbf{b})$. By Corollary 3.3 and Proposition 4.1, it suffices to show that $r(I) + 1 \geq a(\mathbf{f})$, i.e. $((f_1, \dots, f_{i-1})\colon f_i)_n = (f_1, \dots, f_{i-1})_n$ for $n \geq r(I) + 1$, $i = 1, \dots, r$. Translating this condition in terms of $x_1, \dots, x_r$, we have to show that

$$[((x_1, \dots, x_{i-1})\mathbf{a}^n + \mathbf{a}^{n+2})\colon x_i] \cap \mathbf{a}^n = (x_1, \dots, x_{i-1})\mathbf{a}^{n-1} + \mathbf{a}^{n+1}.$$

Since the left side obviously contains the right side, it is sufficient to prove the converse inclusion. Let x be an arbitrary element of the left side. Then $x_ix = y$ modulo $(x_1, \dots, x_{i-1})\mathbf{a}^n$ for some element $y \in (x_1, \dots, x_i) \cap \mathbf{a}^{n+2} = (x_1, \dots, x_{i-1})\mathbf{a}^{n+1}$.

Thus, there is an element $z \in \mathbf{a}^{n+1}$ such that $x_i(x - z) \in (x_1, \ldots, x_{i-1})$. Since $\mathrm{ht}(\mathbf{b}) = \mathrm{ht}(\mathbf{a}) = r$, $x_1, \ldots, x_r$ belong to a system of parameters of A. Therefore, $x_1, \ldots, x_r$ is a d-sequence [**8**]. Hence $((x_1, \ldots, x_{i-1})\colon x_i) \cap \mathbf{b} = (x_1, \ldots, x_{i-1})$. Note that $\mathbf{a}^n = \mathbf{b}\mathbf{a}^{n-1} \subseteq \mathbf{b}$. Then

$$\begin{aligned} x - z \in ((x_1, \ldots, x_{i-1})\colon x_i) \cap \mathbf{a}^n &= (x_1, \ldots, x_{i-1}) \cap \mathbf{a}^n \\ &= (x_1, \ldots, x_{i-1})\mathbf{a}^{n-1}. \end{aligned}$$

Hence we can conclude that $x \in (x_1, \ldots, x_{i-1})\mathbf{a}^{n-1} + \mathbf{a}^{n+1}$.

Now we will apply Proposition 5.1 to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Without restriction, we may assume that $k = A/\mathbf{m}$ is infinite. Since $\mathbf{a}$ is an $\mathbf{m}$-primary ideal, $\mathbf{b}$ is generated by a system of parameters $x_1, \ldots, x_d$ of A. Set $R = G_{\mathbf{a}}(A)$. By Lemma 3.1, we may assume that the initial forms $f_1, \ldots, f_d$ of $x_1, \ldots, x_d$ in R form a filter-regular sequence. Since $\mathrm{depth}(R) \geq d - 1$, $f_1, \ldots, f_{d-1}$ is a regular sequence. By [**17**], this implies the relation

$$(x_1, \ldots, x_{i-1}) \cap \mathbf{a}^{n+1} = (x_1, \ldots, x_{i-1})\mathbf{a}^n$$

for all $n \geq 0$, $i = 1, \ldots, d - 1$. Hence (i) and (ii) follow from Lemma 5.1. For (iii) we first note that the case $d = 1$ is already known [**1**, Propositions 2.2 and 2.3]. If $d > 1$, $R/f_1R = g_{\mathbf{a}/x_1A}(A/x_1A)$ because f_1 is a non-zero-divisor of R. Thus, $\mathrm{depth}(G_{\mathbf{a}/x_1A}(A/x_1A)) \geq d - 2$ and $r(\mathbf{b}/x_1A) = r(I/f_1R) = r(I) = r(\mathbf{b})$, where $I = (f_1, \ldots, f_d)$. By the induction hypothesis,

$$r(\mathbf{b}) \leq e(A/x_1A)t^{d-2} \quad (\text{resp. } e(A/x_1A)t^{d-2} - 1).$$

On the other hand, like in [**11**, Lemma 1.1], we can choose x_1 so that the initial form g of x_1 in $G_{\mathbf{m}}(A)$ is filter-regular. From this it follows that $(0 : g)_n = 0$ for n sufficiently large. Let s be the degree of g in $G_{\mathbf{m}}(A)$. Then $(\mathbf{m}^{n+s+1}\colon x_1) \cap \mathbf{m}^n = \mathbf{m}^{n+1}$ for n sufficiently large, too. Therefore, x_1 is a superficial element of order s for $\mathbf{m}$ and we get $e(A/x_1A) = e(A)s$ [**19**, Chapter VIII, §8]. Since $x_1 \notin \mathbf{m}\mathbf{a} \subseteq \mathbf{m}^{t+1}$, $s \leq t$. Hence we can conclude that

$$r(\mathbf{b}) \leq e(A)t^{d-1} \quad (\text{resp. } e(A)t^{d-1} - 1).$$

REMARK 5.2. In [**2**, Theorem 2], Cho claimed that if $\mathbf{a}$ is an $\mathbf{m}$-primary ideal of a Cohen-Macaulay ring A such that $R = G_{\mathbf{a}}(A)$ is free over $A/\mathbf{a}$ and $\mathrm{depth}(R) \geq d - 1$, then $m_R \leq e(A)$. But his proof for the case $d > 1$ contains the wrong argument that there is an element $x_1 \in \mathbf{a}$ such that $e(A/x_1A) = e(A)$. For example, that is impossible if $\mathbf{a} \subseteq \mathbf{m}^2$. However, the author of the present paper could neither give another proof nor find a counterexample to this claim. The condition $G_{\mathbf{a}}(A)$ being free over $A/\mathbf{a}$ seems to be very strong. For example, if A is regular, this condition implies that $\mathbf{a}$ is generated by a regular sequence [**18**].

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