

ITERATIONS OF HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

C. J. NEUGEBAUER

ABSTRACT. It is shown that the iteration of two maximal functions is essentially no larger than the inner maximal function.

1. For a measurable $f: \mathbf{R}^n \rightarrow \mathbf{R}$ we let $\lambda_f(y) = |\{x: |f(x)| > y\}|$, $y > 0$, and define

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty \lambda_f(y)^{q/p} d(y^q) \right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{y>0} y \lambda_f(y)^{1/p}, & q = \infty. \end{cases}$$

This is the familiar Lorentz space $L(p, q)$ "norm" of f [4] written in terms of the distribution function λ_f rather than the nonincreasing rearrangement of f . It is well known [4] that $\|f\|_{p,p} = \|f\|_p$, the L^p -norm of f , $\|f\|_{p,q_2} \leq \|f\|_{p,q_1}$, $q_2 \geq q_1$, and $\|f + g\|_{p,q} \leq 2(\|f\|_{p,q} + \|g\|_{p,q})$. This paper is concerned with maximal operators of the form

$$M_{p,q}f(x) = \sup_{x \in Q} \frac{\|f\chi_Q\|_{p,q}}{|Q|^{1/p}},$$

where the sup is extended over all cubes Q with $x \in Q$. In particular, we will study iterations $M_{p,q}M_{r,s}f$ and we will obtain inequalities of the type $M_{p,q}M_{r,s}f(x) \leq CM_{r,s}f(x)$ under certain restrictions on the indices. Special cases of this have been studied extensively. For example, Coifman and Rochberg [3] proved $MM_r f \leq CM_r f$, $1 < r < \infty$, where $M = M_{1,1}$, the usual Hardy-Littlewood maximal function, and $M_r = M_{r,r}$, thus showing that $M_r f \in A_1$, Muckenhoupt's weight class A_1 [6]. More recently, Bruna and Korenblum [1, 2] have established $M_{1,\infty}Mf(x) \leq CMf(x)$, and among other results, used this to show that $M_{1,\infty}T^*f \leq C(T^*f + Mf)$, where T^* is the maximal Calderón-Zygmund singular integral operator.

The operator $M_{p,q}$ has also been studied by Stein [7] who for $q \leq p$ established the weak-type inequality $|\{x: M_{p,q}f(x) > y\}| \leq C(\|f\|_{p,q}/y)^p$. A slightly more general result of this kind can also be found in [5].

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2. We will restrict ourselves to dyadic maximal functions, and at the end of this paper we will make a remark about the nondyadic case.

Let Δ be the collection of dyadic cubes Q in \mathbf{R}^n (see [8]), and define $M_{p,q}f(x) \equiv M_{p,q}^\Delta f(x)$ as before except that the sup is taken over all $Q \in \Delta$ with $x \in Q$.

THEOREM 1. *Let $1 \leq p < r < \infty$, $1 \leq s \leq r$, and $1 \leq q < \infty$. Then $M_{p,q}M_{r,s}f(x) \leq CM_{r,s}f(x)$, $x \in \mathbf{R}^n$, where $C = 2^{1/q} (p/(r-p))^{p/rq}$.*

PROOF. Fix $x_0 \in \mathbf{R}^n$, and let $\tau < M_{p,q}M_{r,s}f(x_0)$. Then there exists $x_0 \in Q \in \Delta$ such that

$$\tau^q < \frac{1}{|Q|^{q/p}} \int_0^\infty \lambda_{(M_{r,s}f) \cdot \chi_Q}(y)^{q/p} d(y^q) = \frac{1}{|Q|^{q/p}} \left(\int_0^A + \int_A^\infty \right).$$

We will estimate both integrals. Since $\lambda_{(M_{r,s}f) \cdot \chi_Q}(y) \leq |Q|$, the first integral $(1/|Q|^{q/p}) \int_0^A \leq A^q$.

To estimate the second integral, let $E_y = \{x: M_{r,s}f(x) > y\}$. Then $E_y = \cup Q_j$, $Q_j \in \Delta$, $Q_i \cap Q_j = \emptyset$, $i \neq j$, and

$$y < \frac{1}{|Q_j|^{1/r}} \left(\int_0^\infty \lambda_{f \chi_{Q_j}}(t)^{s/r} d(t^s) \right)^{1/s}.$$

Hence

$$\frac{|E_y \cap Q|}{|Q|} = \frac{1}{|Q|} \sum |Q_j \cap Q|.$$

Since we deal with the dyadic case, if $Q_j \cap Q \neq \emptyset$, then either $Q_j \subset Q$ or $Q_j \supset Q$. In the second case let $Q = Q_j$ without decreasing $|E_y \cap Q|/|Q|$. Thus we may suppose that $E_y \cap Q = \cup_{Q_j \subset Q} Q_j$, and hence, since

$$|Q_j| < \frac{1}{y^r} \left(\int_0^\infty \lambda_{f \chi_{Q_j}}(t)^{s/r} d(t^s) \right)^{r/s},$$

we get

$$\begin{aligned} |E_y \cap Q| &\leq \sum_{Q_j \subset Q} |Q_j| \leq \frac{1}{y^r} \sum_{Q_j \subset Q} (\)^{r/s} \\ &\leq \frac{1}{y^r} \left\{ \int_0^\infty \left[\sum_j \lambda_{f \cdot \chi_{Q_j}}(t) \right]^{s/r} d(t^s) \right\}^{r/s} \quad (\text{since } r \geq s) \\ &\leq \frac{1}{y^r} \left\{ \int_0^\infty \lambda_{f \chi_Q}(t)^{s/r} d(t^s) \right\}^{r/s}. \end{aligned}$$

From this we obtain

$$\begin{aligned} \frac{\lambda_{(M_{r,s}f) \chi_Q}(y)}{|Q|} &= \frac{|E_y \cap Q|}{|Q|} \leq \frac{1}{|Q|} \frac{1}{y^r} \left(\int_0^\infty \lambda_{f \chi_Q}(y)^{s/r} d(t^s) \right)^{r/s} \\ &\leq \frac{1}{y^r} M_{r,s}f(x_0)^r. \end{aligned}$$

Up to here we only used the restriction $r \geq s$. Since $r > p$ we get

$$\frac{1}{|Q|^{q/p}} \int_A^\infty \leq M_{r,s} f(x_0)^{qr/p} q \int_A^\infty y^{q(1-r/p)-1} dy = M_{r,s} f(x_0)^{qr/p} \frac{P}{r-p} A^{q-qr/p}.$$

We combine the two estimates and obtain

$$\tau^q \leq A^q + \frac{P}{r-p} M_{r,s} f(x_0)^{qr/p} A^{q-qr/p}.$$

If we choose A so that the two terms on the right are equal, i.e., $A = M_{r,s} f(x_0) (p/(r-p))^{p/qr}$, we obtain $\tau^q \leq 2A^q$ or $\tau \leq 2^{1/q} A$, and the proof is complete.

3. This paragraph is concerned with the case $q = \infty$ of Theorem 1.

THEOREM 2. *Let $1 \leq p \leq r < \infty$, $1 \leq s \leq r$. Then $M_{p,\infty} M_{r,s} f(x) \leq M_{r,s} f(x)$, $x \in \mathbf{R}^n$.*

PROOF. If $r > p$, we get from Theorem 1 that

$$M_{p,\infty} M_{r,s} f(x) \leq M_{p,q} M_{r,s} f(x) \leq c_q M_{r,s} f(x)$$

for $q < \infty$, where $c_q = 2^{1/q} (p/(r-p))^{p/rq}$. Hence $c_q \rightarrow 1$ as $q \rightarrow \infty$, and this is the $r > p$ case of the theorem.

We assume now $r = p$. Fix $x_0 \in \mathbf{R}^n$ and let $\tau < M_{p,\infty} M_{p,s} f(x_0)$. Then there is $Q \in \Delta$, $x_0 \in Q$, and there is $y > 0$ such that

$$\tau < \frac{1}{|Q|^{1/p}} y \lambda_{(M_{p,s} f)_{x_0}}(y)^{1/p}.$$

As in Theorem 1, $\lambda_{(M_{p,s} f)_{x_0}}(y)/|Q| \leq (1/y^p) M_{p,s} f(x_0)^p$, and hence $\tau < M_{p,s} f(x_0)$.

COROLLARY 1. *Under the hypothesis of Theorem 2, $M_{p,\infty} M_{r,s} f(x) = M_{r,s} f(x)$ a.e. $x \in \mathbf{R}^n$.*

PROOF. At every point x of approximate continuity of a nonnegative measurable function ϕ , $M_{p,\infty} \phi(x) \geq \phi(x)$.

REMARKS. (i) The restriction $r \geq p$ in Theorem 2 is necessary. As an example, let $f(x) = x^{-2/3} \chi_{[0,1]}(x)$, and let $r = s = 1$, $p = 2$, $q = \infty$. For $0 < x < 1$, $Mf(x) \approx f(x)$, $M = M_{11}$, but the $L(2, \infty)$ norm of f is infinite.

(ii) The restriction $r \geq s$ is also necessary. An example for $p = r = 1$, $q = s = \infty$ can be found in [1].

4. We will now study the nondyadic version of the above results, and see that they are still valid with an additional constant depending upon n . However, one cannot expect an equality as in Corollary 1.

We retain the same notation as before and let $M_{p,q} f(x)$ be the above maximal function where the sup is now extended over all cubes Q with $x \in Q$.

THEOREM 3. *If $1 \leq p < r < \infty$, $1 \leq s \leq r$, and $1 \leq q < \infty$, then $M_{p,q} M_{r,s} f(x) \leq c M_{r,s} f(x)$, $x \in \mathbf{R}^n$, where c depends only upon n , r , p , q .*

PROOF. Fix $x_0 \in \mathbf{R}^n$, and let $\tau < M_{p,q} M_{r,s} f(x_0)$. As in the proof of Theorem 1, there is $x_0 \in Q$ such that

$$\tau^q \leq \frac{1}{|Q|^{q/p}} \left(\int_0^A + \int_A^\infty \right) \lambda_{(M_{r,s}f)\chi_Q}(y)^{q/p} d(y^q),$$

and again $(1/|Q|^{q/p}) \int_0^A \leq A^q$.

We now use a familiar technique [3] and break up f as $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{\mathbf{R}^n \setminus 2Q}$. Since $M_{r,s}f(x) \leq 2(M_{r,s}f_1(x) + M_{r,s}f_2(x))$, we see that

$$\lambda_{M_{r,s}f}(y) \leq \lambda_{M_{r,s}f_1}\left(\frac{y}{4}\right) + \lambda_{M_{r,s}f_2}\left(\frac{y}{4}\right).$$

Hence the integral \int_A^∞ is majorized by two integrals, one with f_1 , the other with f_2 .

For the integral with f_1 , let $E_y = \{x: M_{r,s}f_1(x) > y\}$. Then there is a disjoint collection of cubes $\{Q_j\}$ such that $|E_y| \leq c_n \sum |Q_j|$ and

$$y < \left(1/|Q_j|^{1/r}\right) \left(\int_0^\infty \lambda_{f_1\chi_{Q_j}}(t)^{s/r} d(t^s)\right)^{1/s}.$$

As in the proof of Theorem 1,

$$|E_y| \leq c \frac{1}{y^r} \sum (\)^{r/s} \leq c \frac{1}{y^r} \left\{ \int_0^\infty \lambda_{f_1\chi_{\cup Q_j}}(t)^{s/r} d(t^s) \right\}^{r/s}.$$

From this we get that

$$\begin{aligned} \frac{\lambda_{(M_{r,s}f_1)\chi_Q}(y)}{|Q|} &\leq \frac{c}{|Q|} \frac{1}{y^r} \left(\int_0^\infty \lambda_{f_1\chi_{2Q}}(y)^{s/r} d(t^s) \right)^{r/s} \\ &\leq \frac{c}{y^r} M_{r,s}f(x_0)^r. \end{aligned}$$

The integral $\int_A^\infty \lambda_{(M_{r,s}f_2)\chi_Q}(y/4)^{q/p} d(y^q)$ is easily disposed of. First we note that $M_{r,s}f_2$ is essentially constant on Q , i.e., $cM_{r,s}f_2(x_0) \geq M_{r,s}f_2(x)$, $x \in Q$, with $c \approx 5^{n/r}$. If $E_y = \{x: M_{r,s}f_2(x) > y\}$, then

$$|E_y \cap Q| \leq |\{x \in Q: cM_{r,s}f_2(x_0) \geq y\}| \leq c^r |Q| \left(\frac{M_{r,s}f_2(x_0)}{y} \right)^r.$$

Hence again

$$\frac{1}{|Q|} \lambda_{(M_{r,s}f_2)\chi_Q}(y) \leq \frac{c^r}{y^r} M_{r,s}f_2(x_0)^r.$$

The proof can now be completed as in Theorem 1.

COROLLARY 2. If $1 < r < \infty$ and $1 \leq s \leq r$, then $M_{r,s}f \in A_1$.

PROOF. This follows from Theorem 4 with $p = q = 1$.

COROLLARY 3. With the same hypothesis as Theorem 3, $M_{p,q} M_{r,s}f \in A_1$.

PROOF. If $M = M_{1,1}$, then $MM_{p,q} M_{r,s}f \leq C_1 MM_{r,s}f \leq C_2 M_{r,s}f \leq C_2 M_{p,q} M_{r,s}f$.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907