

CLOSEDNESS OF INDEX VALUES FOR SUBFACTORS

DAVID HANDELMAN AND HANS WENZL

ABSTRACT. Let $\{A_i \subset B_i\}$ be a collection of inclusions of finite factors with the indices $\{[B_i : A_i]\}$ bounded and each with trivial relative commutant. Then any W^* ultraproduct yields an inclusion with trivial relative commutant whose index is the ultralimit of $\{[B_i : A_i]\}$. In particular, the set of values of indices arising from pairs of factors with trivial relative commutant is a closed subset of \mathbf{R}^+ .

1. The main result of [J] asserts that the index of a subfactor of a II_1 factor has to be in the set

$$I = \{4 \cos^2(\pi/n) \mid n = 3, 4, \dots\} \cup \{x \geq 4 \mid x \in \mathbf{R}\}.$$

However, there is still not much known about the set I_t of index values for subfactors with trivial relative commutant. All values obtained so far in I_t are algebraic integers; moreover, they are the largest eigenvalues of matrices of the form $A'A$ for special nonnegative integer square matrices. (See also the discussion at the end of this section, or [W].)

It had already been shown by Jones that I_t is closed under multiplication. We will show among other things that it is topologically closed.

As indicated in the abstract we prove the closedness of I_t by showing that ultraproducts preserve the trivial relative commutant property and that the index behaves as it should.

Let $\{R_i\}_{i \in \mathbf{N}}$ be a family of finite W^* factors with traces $\{\tau_i\}$, and let \mathcal{F} be a nonprincipal (\equiv free) ultrafilter on \mathbf{N} . Define

$$R = l^\infty(R_i) = \{r = (r_i) \mid r_i \in R_i, \sup \|r_i\| < \infty\},$$

$$M(\mathcal{F}, R) = \{r \in R \mid \forall \text{ positive real } \varepsilon, \{i \in \mathbf{N} \mid \tau_i(r_i r_i^*) < \varepsilon\} \in \mathcal{F}\}.$$

Then, as is well known, R is a W^* algebra, $M(\mathcal{F}, R)$ is a norm-closed ideal of R , and R_∞ , defined as $R/M(\mathcal{F}, R)$, is a finite factor. Now let $\{A_i \subset B_i\}$ be a family of inclusions of W^* factors. Taking the same ultrafilter \mathcal{F} on \mathbf{N} , we construct $A_\infty \rightarrow B_\infty$. However, we may also construct the ultraproduct of $\{B_i\}$ as right A_i -modules; this is $B/M(\mathcal{F}, A) \cdot B$.

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To compute the index $[B_\infty : A_\infty]$, we are going to use the dimension of B_∞ as a (finitely generated projective) A_∞ -module, as in Pimsner and Popa [PP]. For this reason we must show that the ultraproduct of the $\{B_i\}$ as an A_∞ -module is (naturally) the same as the ultraproduct of the B_i as W^* -factors. In other words, we require

$$\mathcal{M}(\mathcal{F}, A)B = \mathcal{M}(\mathcal{F}, B)$$

(notice that one inclusion is obvious). This is a routine deduction from results in [PP].

LEMMA 1. *With the notation as above,*

$$\mathcal{M}(\mathcal{F}, A)B = \mathcal{M}(\mathcal{F}, B)$$

provided $\sup[B_i : A_i] < \infty$.

PROOF. Since $\mathcal{M}(\mathcal{F}, B)$ is a two-sided ideal of B , the inclusion of the left side in the right is clear. Let n be an integer exceeding $[B_i : A_i]$ for all i . For each i , there exists a set of n elements of B_i , $\{m_j^i\}_{j=1}^n$, with the properties ascribed to them in [PP, I.3].

Since for i fixed, $\sum_j (m_j^i)^* m_j^i = [B_i : A_i] < n$, we deduce that each of the sequences $M_j = (m_j^i)_{i \in \mathbb{N}}$ is bounded, so belongs to $l^\infty(B_i) = B$. Select $s = (s_i)$ in $\mathcal{M}(\mathcal{F}, B)$. Write $s_i = \sum_{j=1}^n m_j^i y_{ji}$ for some y_{ji} in A_i . Then $s_i^* s_i = \sum y_{ji}^* (m_j^i)^* m_k^i y_{ki}$.

Let $E_i: B_i \rightarrow A_i$ be the (onto) expectation, and define the projections $e_j^i = E_i((m_j^i)^* m_j^i)$ in A_i . Applying E_i to $s_i^* s_i$ we deduce (as $E_i((m_j^i)^* m_k^i) = 0$ if $j \neq k$), $E(s_i^* s_i) = \sum_j y_{ji}^* e_j^i y_{ji}$. If $e_j^i \neq 1$, we may replace y_{ji} with $e_j^i y_{ji}$, by [PP, I.3(4)]. This guarantees that for all j , $\text{tr}(y_{ji}^* y_{ji}) \leq \text{tr}(s_i s_i^*)$ and similarly with the norm. Set $Y_j = (y_{ij})_{i \in \mathbb{N}}$; as $\|Y_j\| \leq \sup \|s_i\|$, Y_j belongs to $l^\infty(A_i) = A$. The tracial inequality yields that each Y_i belongs to $\mathcal{M}(\mathcal{F}, A)$. Thus

$$s = \sum M_j Y_j \in B\mathcal{M}(\mathcal{F}, A).$$

As $\mathcal{M}(\mathcal{F}, A), \mathcal{M}(\mathcal{F}, B)$ are stable with respect to $*$, s belongs to $\mathcal{M}(\mathcal{F}, A)B$ as desired. \square

Thus the induced map $A_\infty \rightarrow B_\infty$ is not only one-to-one, but (with notation) from the proof above the $\{M_j\}$ generate B_∞ as a right A_∞ -module, so B_∞ is at most n -generated as an A_∞ -module. It follows easily that B_∞ is projective as an A_∞ -module.

By [PP, §1], the index is just the dimension of B_∞ (as a projective A_∞ -module). If $p = (p_i)$ is a sequence of projections in $M_n A = l^\infty(M_n A_i)$, then $\tau_\infty(p) = \lim_{\mathcal{F}} \tau_i(p_i)$, where τ_∞ is the trace on A , thus on $M_n A$, obtained from A_∞ (as in [G2] — this may also be deduced from comparability in $M_n A_\infty$). As an A_i -module, B_i may be regarded as $P_i M_n A_i$ for some projection, and it follows that

$$[B_\infty : A_\infty] = \lim[B_i : A_i]. \quad \square$$

Next, we determine the relative commutant of A_∞ in B_∞ . The following is presumably well known, but we could find no direct reference for it.

LEMMA 2. Let R be a unital C^* -subalgebra of a finite type II factor S , having τ as its trace. Let $\|\cdot\|_2$ denote the tracial 2-norm, on S . Suppose s is an element of S such that for some real $\delta > 0$,

$$\|sr - rs\|_2 < \delta \quad \text{for all } r \text{ in the unit ball of } R.$$

Then there exists z in $R' \cap S (= \{t \in S \mid tr = rt \text{ all } r \text{ in } R\})$ such that $\|s - z\|_2 \leq \delta$.

PROOF. Let C be the $\|\cdot\|_2$ -closure of the convex hull of $\{usu^* \mid u \text{ unitary in } R\}$. This is also strongly closed (as the set is bounded) and thus weakly closed (as the set is convex), so weakly compact. The group of unitaries of R acts affinely on C , so the Ryll-Nardzewski theorem applies (e.g. [Gr, §3.1]). Thus there is a fixed point z in C ; z belongs to $R' \cap S$.

If $z_0 = \sum \lambda_i u_i s u_i^*$ with u_i unitaries in R , λ_i positive real numbers with $\sum \lambda_i = 1$, then

$$\begin{aligned} \|s - z_0\|_2 &= \left\| \sum \lambda_i (s - u_i s u_i^*) \right\|_2 \leq \sum \lambda_i \|s - u_i s u_i^*\|_2 \\ &= \sum \lambda_i \|(s u_i - u_i s) u_i^*\|_2 < \sum \lambda_i \delta = \delta. \end{aligned}$$

As z is the $\|\cdot\|_2$ -closure of such z_0 's, $\|s - z\|_2 \leq \delta$. \square

Returning to our initial assumption that $\sup[B_i : A_i] < n$, we see that $d_i = \dim_C A'_i \cap B_i$ are all less than n . Because d_i can vary over only finitely many values, there exists a unique integer $d < n$ with $\{i \in \mathbf{N} \mid d_i = d\} \in \mathcal{F}$. We now show that $A'_\infty \cap B_\infty$ has dimension d . In particular, if the relative commutants of A_i in B_i are trivial (i.e., $d_i = 1$ for all i) the same is true for that of A_∞ in B_∞ .

Select b^0 in $A'_\infty \cap B_\infty$. There exists $b = (b_i) \in B$ so that $b + \mathcal{M}(\mathcal{F}, B) = b^0$ and $\|b\| = \|b^0\|$. For each ϵ , the set

$$I_\epsilon = \left\{ i \in \mathbf{N} \mid \|r_i b_i - b_i r_i\|_2 < \epsilon \text{ for all } r_i \text{ in the unit ball of } A_i \right\}$$

belongs to \mathcal{F} . For each i in I_ϵ , Lemma 2 yields the existence of z_i in $A'_i \cap B_i$ with $\|b_i - z_i\|_2 \leq \epsilon$, and we may assume z_i is in the unit ball. Thus $Z_\epsilon = (z_i) + \mathcal{M}(\mathcal{F}, B)$ belongs to $A'_\infty \cap B_\infty$ and $\|Z_\epsilon - b^0\|_2 \leq \epsilon$. Thus $A'_\infty \cap B_\infty$ is the closure of the image of $l^\infty(A'_i \cap B_i)$. As $[B_\infty : A_\infty] \leq n$, $A'_\infty \cap B_\infty$ is of finite dimension, so $l^\infty(A'_i \cap B_i)$ maps onto $A'_\infty \cap B_\infty$. It easily follows that $\dim A'_\infty \cap B_\infty = d$ (since if $d_i \neq d$, the i th entries may be discarded). A bit more is true: For each dimension d , there are only finitely many isomorphism classes of C^* -algebras of that dimension—hence one class has to occur at the indices constituting a member of \mathcal{F} (while the others do not). So $A'_\infty \cap B_\infty$ is isomorphic to one of the finite dimensional algebras, $A'_i \cap B_i$. To summarize, we have the following.

THEOREM. Let $\{A_i \subset B_i\}_{i \in \mathbf{N}}$ be a collection of pairs of finite W^* -factors with $\sup[B_i : A_i] < \infty$. Let \mathcal{F} be a nonprincipal ultrafilter on \mathbf{N} , and $A_\infty \subset B_\infty$ the corresponding inclusions of the W^* -ultraproducts. Then

- (i) $[B_\infty : A_\infty] = \lim_{\mathcal{F}} [B_i : A_i]$.
- (ii) $l^\infty(A'_i \cap B_i)$ maps onto $A'_\infty \cap B_\infty$.
- (iii) $\{i \in \mathbf{N} \mid A'_i \cap B_i \cong A'_\infty \cap B_\infty\} \in \mathcal{F}$.

In particular, if $A'_i \cap B_i = \mathbf{C}$ for all i , then $A'_\infty \cap B_\infty = \mathbf{C}$. If $\lambda = \lim_i [B_i : A_i]$ exists, then $[B_\infty : A_\infty] = \lambda$.

COROLLARY. Let D be a finite dimensional C^* algebra. Then the subset of \mathbf{R}^+ given by

$$\{ \lambda \mid \text{there exists a pair of finite type } W^*\text{-factors } A \subset B \\ \text{with } [B : A] = \lambda \text{ and } A' \cap B \cong D \}$$

is closed in \mathbf{R} .

Of course, the only interesting case is if $A' \cap B = \mathbf{C}$, when the set in the Corollary is I_r . Unfortunately, this does not produce any new values in it. We can however rule out one conjecture about I_r , namely that it consists precisely of all largest eigenvalues of matrices of the form $A'A$, with A having only nonnegative integer entries. Indeed by an unpublished result of James Shearer (communicated to us by A. Hoffman) the spectral radii of incidence matrices of graphs are dense in $[\lambda, \infty)$, where $\lambda = \tau^{1/2} + \tau^{-1/2}$ with τ the golden ratio. As these matrices are symmetric, it follows that the set in this conjecture would be dense in $[\lambda^2, \infty)$. In that case, I_r would contain the entire interval, including transcendentals.

It is well known and obvious that I_r contains \mathbf{N} and by results of [J], $[I_r]$ contains $\{4 \cos^2 \pi/n \mid n = 3, 4, 5, \dots\}$. This was generalized in [W] as follows. Let k, m be positive integers with $1 \leq k \leq m - 2$, and let $\lambda = (\lambda_i)_{i \leq k}$ be a young diagram having at most k rows, λ_i being the length of the i th row. Assume additionally $\lambda_1 - \lambda_k \leq m - k$. Then there exists a subfactor of the hyperfinite II_1 factor R with trivial relative commutant

$$\lambda(m) = \prod_{1 \leq r < s \leq k} \frac{\sin^2((\lambda_r - \lambda_s + s - r)\pi/m)}{\sin^2((s - r)\pi/m)}.$$

If $k = 2$, and $\lambda = (1)$, the index value will be $4 \cos^2 \pi/m$.

It is easy to check that

$$\lim_{m \rightarrow \infty} \lambda(m) = \prod_{1 \leq r < s \leq k} \frac{(\lambda_r - \lambda_s + s - r)^2}{(s - r)^2}$$

which is the square of the dimension of the character of $SU(k)$ with dominant weight $(\lambda_1 - \lambda_k, \lambda_2 - \lambda_k, \dots, \lambda_{k-1} - \lambda_k)$.

2. Alternate paths. Given the essentially routine nature of all the arguments above, it is not surprising that there are many other methods of proof.

For instance, to show as in [PP, I.3] that if $A \subset B$ are finite factors and ${}_A B$ is finitely generated (i.e., as an A -module), then ${}_A B$ is projective follows from [GH, 5.6; and G, 5.18]. To obtain the orthogonal decomposition of [PP, I.3], just apply [BH, III.2.10] to matrices; this yields that any finitely generated closed submodule of a finitely generated projective module over a C^* algebra is a direct summand. These direct summands can be orthogonalized as a result of [K].

We can give a different proof for the triviality of the relative commutant of A_∞ in B_∞ by using the following criterion that is also interesting in its own right.

Let $U(S)$ be the unitary group of S , $l(x)$ and $r(x)$ the left resp. right support of x and, if x is normal let $\text{supp}(x)$ be its support.

LEMMA 3. *Let $R \subset S$, $[S : R] < \infty$, and $\tau = 1/[S : R]$. Then we have for any projection e in R ,*

$$\text{tr}(\text{supp } E_R(e)) \leq [S : R] \text{tr}(e).$$

PROOF. If $\text{tr}(e) \geq \tau$, the statement is trivial. Note that by [J, 3.1.8] there exists a projection f in S such that $E_R(f) = \tau$. Let us just assume $e \leq f$. Then

$$eE_R(e)e \leq eE_R(f)e = \tau e;$$

on the other hand, $eE_R(e)e \geq \tau e$ by [PP, 2.1]. Hence

$$\tau \text{tr}(E_R(e)) = \tau \text{tr}(e) = \text{tr}(eE_R(e)e) = \text{tr}(E_R(e)^2).$$

As $E_R(e) \leq \tau$, it follows from spectral calculus that the only nonzero spectral value of $E_R(e)$ is τ . Hence $E_R(e) = \tau q$ with $q = \text{supp}(E_R(e))$ and obviously $\text{tr}(q) = 1/\tau \text{tr}(e)$. If e' is an arbitrary projection with $\text{tr}(e') \leq \tau$, we can find partial isometries v and v^* such that $vv^* = e'$ and $v^*v = e \leq f$. In particular $e' = vfv^*$. By [PP, 1.2] and its proof, there exists m in R such that $mf = vf$ and $e' = mfm^*$. It follows that $r(mf) = r(vf) = e$ and $e' = mem^*$. Hence $E_R(e') = \tau mqm^*$ and $\text{supp}(E_R(e')) = l(mq) \sim r(mq) \leq q$. \square

PROPOSITION. *Let R, S, τ be as in the lemma. Then the following are equivalent:*

- (i) $R' \cap S = \mathbf{C}$.
- (ii) *For any two projections p, q in S with $p > q$ and $\epsilon > 0$ there exists a unitary u in R such that $\text{tr}(u^*puq) \geq (\tau - \epsilon)\text{tr}(q)$.*

PROOF. (ii) \Rightarrow (i) Assume $R' \cap S \neq \mathbf{C}$ and let e be a nontrivial projection of $R' \cap S$ with $\text{tr}(e) \leq 1/2$. If we set $q = e$ and $p = 1 - e$ we obtain $\text{tr}(u^*puq) = \text{tr}(pq) = 0$ for any u in $U(R)$.

(i) \Rightarrow (ii) Let us first assume $\text{tr}(p) \geq \tau$. It follows from the Ryll-Nardzewski fixed point theorem that $\delta(p)$, the weak closure of the convex hull of $\{upu^* \mid u \in U(R)\}$ contains an element $\tilde{p} \in R' \cap S = \mathbf{C}1$. As tr is constant on $\delta(p)$, $\tilde{p} = \text{tr}(p) \geq \tau$. But if $\text{tr}(upu^*q) \leq (\tau - \epsilon)\text{tr}(q)$ for all $u \in U(R)$, $\text{tr}(p)$ could never be approximated by elements of $\delta(p)$. Note that for any projection $f \in R$, $[S_f : R_f] = [S : R]$ by [J, 2.1.4 and 2.1.7] and $R'_f \cap S_f = (R' \cap S)_f \cong \mathbf{C}$. Let $p_0 = \text{supp } E_R(p)$, $q_0 = \text{supp}(E_R(q))$. As it is enough to show (ii) for any upu^* , u in $U(R)$, instead of p , we can assume either $p_0 \leq q_0$ or $q_0 \leq p_0$. Note in the first case that

$$\text{tr}_{S_{q_0}}(p) = \frac{\text{tr}(p)}{\text{tr}(q_0)} \geq \frac{\text{tr}(q)}{\text{tr}(q_0)} \geq \tau$$

by Lemma 3. Hence we can find an appropriate unitary in R_{q_0} (which can be extended easily to R). Similarly, we can prove the claim for $q_0 \leq p_0$ by reducing by p_0 . \square

The constant in (ii) is also the best we can hope for. Let q be a projection in S such that $E_R(q) = \tau$ and let p be a projection in R such that $\text{tr}(p) = \tau$. Then

$$\text{tr}(upu^*q) = \text{tr}(upu^*(E_R(q))) = \tau \text{tr}(p) = \tau \text{tr}(q)$$

for all $u \in U(R)$.

To show that $A_\infty \subset B_\infty$ has trivial relative commutant, just observe that we can find for projections p and q in B_∞ with $q \leq p$, sequences (p_i) and (q_i) of projections with $\text{tr}_i(q_i) \leq \text{tr}_i(p_i)$ such that $q = \overline{(q_i)}$ and $p = \overline{(p_i)}$. For any $\varepsilon > 0$, we find a unitary u_i in A_i such that (ii) of the Proposition holds. So $U = (u_i) + \mathcal{M}(\mathcal{F}, A_i)$ is a unitary demonstrating (ii) for p and q . \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTTAWA, OTTAWA, ONTARIO K1N 6N5, CANADA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720