PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 2, October 1987

CLOSEDNESS OF INDEX VALUES FOR SUBFACTORS

DAVID HANDELMAN AND HANS WENZL

ABSTRACT. Let $\{A_i \subset B_i\}$ be a collection of inclusions of finite factors with the indices $\{[B_i : A_i]\}$ bounded and each with trivial relative commutant. Then any W^* ultraproduct yields an inclusion with trivial relative commutant whose index is the ultralimit of $\{[B_i : A_i]\}$. In particular, the set of values of indices arising from pairs of factors with trivial relative commutant is a closed subset of $\mathbf{R}^+$.

1. The main result of **[J]** asserts that the index of a subfactor of a II_1 factor has to be in the set

$$I = \{4\cos^2(\pi/n) \mid n = 3,4,\dots\} \cup \{x \geqslant 4 \mid x \in \mathbf{R}\}.$$

However, there is still not much known about the set I_t of index values for subfactors with trivial relative commutant. All values obtained so far in I_t are algebraic integers; moreover, they are the largest eigenvalues of matrices of the form A^tA for special nonnegative integer square matrices. (See also the discussion at the end of this section, or **[W]**.)

It had already been shown by Jones that I_t is closed under multiplication. We will show among other things that it is topologically closed.

As indicated in the abstract we prove the closedness of I_t by showing that ultraproducts preserve the trivial relative commutant property and that the index behaves as it should.

Let $\{R_i\}_{i\in\mathbf{N}}$ be a family of finite W^* factors with traces $\{\tau_i\}$, and let $\mathscr{F}$ be a nonprincipal ($\equiv$ free) ultrafilter on $\mathbf{N}$. Define

$$R = l^\infty(R_i) = \{r = (r_i) \mid r_i \in R_i, \sup\|r_i\| < \infty\},$$

$$M(\mathscr{F}, R) = \{r \in R \mid \forall \text{ positive real } \varepsilon, \{i \in \mathbf{N} \mid \tau_i(r_ir_i^*) < \varepsilon\} \in \mathscr{F}\}.$$

Then, as is well known, R is a W^* algebra, $M(\mathscr{F}, R)$ is a norm-closed ideal of R, and R_∞, defined as $R/\mathscr{M}(\mathscr{F}, R)$, is a finite factor. Now let $\{A_i \subset B_i\}$ be a family of inclusions of W^* factors. Taking the same ultrafilter $\mathscr{F}$ on $\mathbf{N}$, we construct $A_\infty \to B_\infty$. However, we may also construct the ultraproduct of $\{B_i\}$ as right A_i-modules; this is $B/\mathscr{M}(\mathscr{F}, A)\cdot B$.

Received by the editors April 24, 1986 and, in revised form, June 9, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L10.
The first author was supported by an operating grant and a Steacie Memorical Fellowship from NSERC (Canada).
The second author was supported by Handelman's operating grant.

To compute the index $[B_\infty : A_\infty]$, we are going to use the dimension of B_∞ as a (finitely generated projective) A_∞-module, as in Pimsner and Popa **[PP]**. For this reason we must show that the ultraproduct of the $\{B_i\}$ as an A_∞-module is (naturally) the same as the ultraproduct of the B_i as W^*-factors. In other words, we require

$$\mathscr{M}(\mathscr{F}, A)B = \mathscr{M}(\mathscr{F}, B)$$

(notice that one inclusion is obvious). This is a routine deduction from results in **[PP]**.

LEMMA 1. *With the notation as above,*

$$\mathscr{M}(\mathscr{F}, A)B = \mathscr{M}(\mathscr{F}, B)$$

provided $\sup[B_i : A_i] < \infty$.

PROOF. Since $\mathscr{M}(\mathscr{F}, B)$ is a two-sided ideal of B, the inclusion of the left side in the right is clear. Let n be an integer exceeding $[B_i : A_i]$ for all i. For each i, there exists a set of n elements of B_i, $\{m_j^i\}_{j=1}^n$, with the properties ascribed to them in **[PP**, I.3].

Since for i fixed, $\sum_j (m_j^i)^* m_j^i = [B_i : A_i] < n$, we deduce that each of the sequences $M_j = (m_j^i)_{i \in \mathbf{N}}$ is bounded, so belongs to $l^\infty(B_i) = B$. Select $s = (s_i)$ in $\mathscr{M}(\mathscr{F}, B)$. Write $s_i = \sum_{j=1}^n m_j^i y_{j1}$ for some y_{ji} in A_i. Then $s_i^* s_i = \sum y_{ji}^* (m_j^i)^* m_k^i y_{ki}$.

Let E_i: $B_i \to A_i$ be the (onto) expectation, and define the projections $e_j^i = E_i((m_j^i)^* m_j^i)$ in A_i. Applying E_i to $s_i^* s_i$ we deduce (as $E_i((m_j^i)^* m_k) = 0$ if $j \neq k$), $E(s_i^* s_i) = \sum_j y_{ji}^* e_j^i y_{ji}$. If $e_j^i \neq 1$, we may replace y_{ji} with $e_j^i y_{ji}$, by **[PP**, I.3(4)]. This guarantees that for all j, $\operatorname{tr}(y_{ji}^* y_{ji}) \leqslant \operatorname{tr}(s_i s_i^*)$ and similarly with the norm. Set $Y_j = (y_{ij})_{i \in \mathbf{N}}$; as $\|Y_j\| \leqslant \sup \|s_i\|$, Y_j belongs to $l^\infty(A_i) = A$. The tracial inequality yields that each Y_i belongs to $\mathscr{M}(\mathscr{F}, A)$. Thus

$$s = \sum M_j Y_j \in B\mathscr{M}(\mathscr{F}, A).$$

As $\mathscr{M}(\mathscr{F}, A)$, $\mathscr{M}(\mathscr{F}, B)$ are stable with respect to $*$, s belongs to $\mathscr{M}(\mathscr{F}, A)B$ as desired. □

Thus the induced map $A_\infty \to B_\infty$ is not only one-to-one, but (with notation) from the proof above the $\{M_j\}$ generate B_∞ as a right A_∞-module, so B_∞ is at most n-generated as an A_∞-module. It follows easily that B_∞ is projective as an A_∞-module.

By **[PP**, §1], the index is just the dimension of B_∞ (as a projective A_∞-module). If $p = (p_i)$ is a sequence of projections in $M_n A = l^\infty(M_n A_i)$, then $\tau_\infty(p) = \lim_{\mathscr{F}} \tau_i(p_i)$, where τ_∞ is the trace on A, thus on $M_n A$, obtained from A_∞ (as in **[G2]** —this may also be deduced from comparability in $M_n A_\infty$). As an A_i-module, B_i may be regarded as $P_i M_n A_i$ for some projection, and it follows that

$$[B_\infty : A_\infty] = \lim[B_i : A_i]. \qquad \square$$

Next, we determine the relative commutant of A_∞ in B_∞. The following is presumably well known, but we could find no direct reference for it.

LEMMA 2. *Let R be a unital C^*-subalgebra of a finite type* II *factor S, having τ as its trace. Let $\| \ \|_2$ denote the tracial 2-norm, on S. Suppose s is an element of S such that for some real $\delta > 0$,*

$$\|sr - rs\|_2 < \delta \quad \textit{for all } r \textit{ in the unit ball of } R.$$

Then there exists z in $R' \cap S$ $(= \{t \in S \mid tr = rt \text{ all } r \text{ in } R\})$ such that $\|s - z\|_2 \leqslant \delta$.

PROOF. Let C be the $\| \ \|_2$-closure of the convex hull of $\{usu^* \mid u$ unitary in $R\}$. This is also strongly closed (as the set is bounded) and thus weakly closed (as the set is convex), so weakly compact. The group of unitaries of R acts affinely on C, so the Ryll-Nardzewski theorem applies (e.g. [**Gr**, §3.1]). Thus there is a fixed point z in C; z belongs to $R' \cap S$.

If $z_0 = \sum \lambda_i u_i s u_i^*$ with u_i unitaries in R, λ_i positive real numbers with $\sum \lambda_i = 1$, then

$$\begin{aligned}\|s - z_0\|_2 &= \left\| \sum \lambda_i \left(s - u_i s u_i^*\right) \right\|_2 \leqslant \sum \lambda_i \|s - u_i s u_i^*\|_2 \\ &= \sum \lambda_i \|(s u_i - u_i s) u_i^*\|_2 < \sum \lambda_i \delta = \delta.\end{aligned}$$

As z is the $\| \ \|_2$-closure of such z_0's, $\|s - z\| \leqslant \delta$. □

Returning to our initial assumption that $\sup[B_i : A_i] < n$, we see that $d_i = \dim_{\mathbf{C}} A_i' \cap B_i$ are all less than n. Because d_i can vary over only finitely many values, there exists a unique integer $d < n$ with $\{i \in \mathbf{N} \mid d_i = d\} \in \mathscr{F}$. We now show that $A_\infty' \cap B_\infty$ has dimension d. In particular, if the relative commutants of A_i in B_i are trivial (i.e., $d_i = 1$ for all i) the same is true for that of A_∞ in B_∞.

Select b^0 in $A_\infty' \cap B_\infty$. There exists $b = (b_i) \in B$ so that $b + \mathscr{M}(\mathscr{F}, B) = b^0$ and $\|b\| = \|b^0\|$. For each ε, the set

$$I_\varepsilon = \left\{ i \in \mathbf{N} \mid \|r_i b_i - b_i r_i\|_2 < \varepsilon \text{ for all } r_i \text{ in the unit ball of } A_i \right\}$$

belongs to $\mathscr{F}$. For each i in I_ε, Lemma 2 yields the existence of z_i in $A_i' \cap B_i$ with $\|b_i - z_i\|_2 \leqslant \varepsilon$, and we may assume z_i is in the unit ball. Thus $Z_\varepsilon = (z_i) + \mathscr{M}(\mathscr{F}, B)$ belongs to $A_\infty' \cap B_\infty$ and $\|Z_\varepsilon - b^0\|_2 \leqslant \varepsilon$. Thus $A_\infty' \cap B_\infty$ is the closure of the image of $l^\infty(A_i' \cap B_i)$. As $[B_\infty : A_\infty] \leqslant n$, $A_\infty' \cap B_\infty$ is of finite dimension, so $l^\infty(A_i' \cap B_i)$ maps onto $A_\infty' \cap B_\infty$. It easily follows that $\dim A_\infty' \cap B_\infty = d$ (since if $d_i \neq d$, the ith entries may be discarded). A bit more is true: For each dimension d, there are only finitely many isomorphism classes of C^*-algebras of that dimension—hence one class has to occur at the indices constituting a member of $\mathscr{F}$ (while the others do not). So $A_\infty' \cap B_\infty$ is isomorphic to one of the finite dimensional algebras, $A_i' \cap B_i$. To summarize, we have the following.

THEOREM. *Let $\{A_i \subset B_i\}_{i \in \mathbf{N}}$ be a collection of pairs of finite W^*-factors with* $\sup[B_i : A_i] < \infty$. *Let $\mathscr{F}$ be a nonprincipal ultrafilter on* $\mathbf{N}$, *and $A_\infty \subset B_\infty$ the corresponding inclusions of the W^*-ultraproducts. Then*

(i) $[B_\infty : A_\infty] = \lim_{\mathscr{F}} [B_i : A_i]$.

(ii) $l^\infty(A_i' \cap B_i)$ *maps onto* $A_\infty' \cap B_\infty$.

(iii) $\{i \in \mathbf{N} \mid A_i' \cap B_i \simeq A_\infty' \cap B_\infty\} \in \mathscr{F}$.

In particular, if $A_i' \cap B_i = \mathbf{C}$ *for all* i, *then* $A_\infty' \cap B_\infty = \mathbf{C}$. *If* $\lambda = \lim_i [B_i : A_i]$ *exists, then* $[B_\infty : A_\infty] = \lambda$.

COROLLARY. *Let* D *be a finite dimensional* C^* *algebra. Then the subset of* $\mathbf{R}^+$ *given by*

$$\{\lambda \mid \text{there exists a pair of finite type } W^*\text{-factors } A \subset B \text{ with } [B:A] = \lambda \text{ and } A' \cap B \simeq D\}$$

is closed in $\mathbf{R}$.

Of course, the only interesting case is if $A' \cap B = \mathbf{C}$, when the set in the Corollary is I_t. Unfortunately, this does not produce any new values in it. We can however rule out one conjecture about I_t, namely that it consists precisely of all largest eigenvalues of matrices of the form A^tA, with A having only nonnegative integer entries. Indeed by an unpublished result of James Shearer (communicated to us by A. Hoffman) the spectral radii of incidence matrices of graphs are dense in $[\lambda, \infty)$, where $\lambda = \tau^{1/2} + \tau^{-1/2}$ with τ the golden ratio. As these matrices are symmetric, it follows that the set in this conjecture would be dense in $[\lambda^2, \infty)$. In that case, I_t would contain the entire interval, including transcendentals.

It is well known and obvious that I_t contains $\mathbf{N}$ and by results of **[J]**, $[I_t]$ contains $\{4\cos^2 \pi/n \mid n = 3, 4, 5, \dots\}$. This was generalized in **[W]** as follows. Let k, m be positive integers with $1 \leqslant k \leqslant m - 2$, and let $\lambda = (\lambda_i)_{i \leqslant k}$ be a young diagram having at most k rows, λ_i being the length of the ith row. Assume additionally $\lambda_1 - \lambda_k \leqslant m - k$. Then there exists a subfactor of the hyperfinite II_1 factor R with trivial relative commutant

$$\lambda(m) = \prod_{1 \leqslant r < s \leqslant k} \frac{\sin^2((\lambda_r - \lambda_s + s - r)\pi/m)}{\sin^2((s - r)\pi/m)}.$$

If $k = 2$, and $\lambda = (1)$, the index value will be $4\cos^2 \pi/m$.

It is easy to check that

$$\lim_{m \to \infty} \lambda(m) = \prod_{1 \leqslant r < s \leqslant k} \frac{(\lambda_r - \lambda_s + s - r)^2}{(s - r)^2}$$

which is the square of the dimension of the character of $SU(k)$ with dominant weight $(\lambda_1 - \lambda_k, \lambda_2 - \lambda_k, \dots, \lambda_{k-1} - \lambda_k)$.

2. Alternate paths. Given the essentially routine nature of all the arguments above, it is not surprising that there are many other methods of proof.

For instance, to show as in **[PP**, I.3] that if $A \subset B$ are finite factors and ${}_AB$ is finitely generated (i.e., as an A-module), then ${}_AB$ is projective follows from **[GH**, 5.6; and **G**, 5.18]. To obtain the orthogonal decomposition of **[PP**, I.3], just apply **[BH**, III.2.10] to matrices; this yields that any finitely generated closed submodule of a finitely generated projective module over a C^* algebra is a direct summand. These direct summands can be orthogonalized as a result of **[K]**.

We can give a different proof for the triviality of the relative commutant of A_∞ in B_∞ by using the following criterion that is also interesting in its own right.

Let $U(S)$ be the unitary group of S, $l(x)$ and $r(x)$ the left resp. right support of x and, if x is normal let $\mathrm{supp}(x)$ be its support.

LEMMA 3. *Let $R \subset S$, $[S:R] < \infty$, and $\tau = 1/[S:R]$. Then we have for any projection e in R,*

$$\mathrm{tr}(\mathrm{supp}\, E_R(e)) \leqslant [S:R]\,\mathrm{tr}(e).$$

PROOF. If $\mathrm{tr}(e) \geqslant \tau$, the statement is trivial. Note that by [**J**, 3.1.8] there exists a projection f in S such that $E_R(f) = \tau$. Let us just assume $e \leqslant f$. Then

$$eE_R(e)e \leqslant eE_R(f)e = \tau e;$$

on the other hand, $eE_R(e)e \geqslant \tau e$ by [**PP**, 2.1]. Hence

$$\tau\,\mathrm{tr}(E_R(e)) = \tau\,\mathrm{tr}(e) = \mathrm{tr}(eE_R(e)e) = \mathrm{tr}\left(E_R(e)^2\right).$$

As $E_R(e) \leqslant \tau$, it follows from spectral calculus that the only nonzero spectral value of $E_R(e)$ is τ. Hence $E_R(e) = \tau q$ with $q = \mathrm{supp}(E_R(e))$ and obviously $\mathrm{tr}(q) = 1/\tau\,\mathrm{tr}(e)$. If e' is an arbitrary projection with $\mathrm{tr}(e') \leqslant \tau$, we can find partial isometries v and v^* such that $vv^* = e'$ and $v^*v = e \leqslant f$. In particular $e' = vfv^*$. By [**PP**, 1.2] and its proof, there exists m in R such that $mf = vf$ and $e' = mfm^*$. It follows that $r(mf) = r(vf) = e$ and $e' = mem^*$. Hence $E_R(e') = \tau mqm^*$ and $\mathrm{supp}(E_R(e')) = l(mq) \sim r(mq) \leqslant q$. □

PROPOSITION. *Let R, S, τ be as in the lemma. Then the following are equivalent*:

(i) $R' \cap S = \mathbf{C}$.

(ii) *For any two projections p, q in S with $p > q$ and $\varepsilon > 0$ there exists a unitary u in R such that* $\mathrm{tr}(u^*puq) \geqslant (\tau - \varepsilon)\,\mathrm{tr}(q)$.

PROOF. (ii) $\Rightarrow$ (i) Assume $R' \cap S \neq \mathbf{C}$ and let e be a nontrivial projection of $R' \cap S$ with $\mathrm{tr}(e) \leqslant 1/2$. If we set $q = e$ and $p = 1 - e$ we obtain $\mathrm{tr}(u^*puq) = \mathrm{tr}(pq) = 0$ for any u in $U(R)$.

(i) $\Rightarrow$ (ii) Let us first assume $\mathrm{tr}(p) \geqslant \tau$. It follows from the Ryll-Nardzewski fixed point theorem that $\delta(p)$, the weak closure of the convex hull of $\{upu^* \mid u \in U(R)\}$ contains an element $\tilde{p} \in R' \cap S = \mathbf{C}1$. As tr is constant on $\delta(p)$, $\tilde{p} = \mathrm{tr}(p) \geqslant \tau$. But if $\mathrm{tr}(upu^*q) \leqslant (\tau - \varepsilon)\,\mathrm{tr}(q)$ for all $u \in U(R)$, $\mathrm{tr}(p)$ could never be approximated by elements of $\delta(p)$. Note that for any projection $f \in R$, $[S_f : R_f] = [S:R]$ by [**J**, 2.1.4 and 2.1.7] and $R'_f \cap S_f = (R' \cap S)_f \cong \mathbf{C}$. Let $p_0 = \mathrm{supp}\, E_R(p)$, $q_0 = \mathrm{supp}(E_R(q))$. As it is enough to show (ii) for any upu^*, u in $U(R)$, instead of p, we can assume either $p_0 \leqslant q_0$ or $q_0 \leqslant p_0$. Note in the first case that

$$\mathrm{tr}_{S_{q_0}}(p) = \frac{\mathrm{tr}(p)}{\mathrm{tr}(q_0)} \geqslant \frac{\mathrm{tr}(q)}{\mathrm{tr}(q_0)} \geqslant \tau$$

by Lemma 3. Hence we can find an appropriate unitary in R_{q_0} (which can be extended easily to R). Similarly, we can prove the claim for $q_0 \leqslant p_0$ by reducing by p_0. □

The constant in (ii) is also the best we can hope for. Let q be a projection in S such that $E_R(q) = \tau$ and let p be a projection in R such that $\mathrm{tr}(p) = \tau$. Then

$$\mathrm{tr}(upu^*q) \doteq \mathrm{tr}(upu^*(E_R(q))) = \tau\,\mathrm{tr}(p) = \tau\,\mathrm{tr}(q)$$

for all $u \in U(R)$.

To show that $A_\infty \subset B_\infty$ has trivial relative commutant, just observe that we can find for projections p and q in B_∞ with $q \lesssim p$, sequences (p_i) and (q_i) of projections with $\mathrm{tr}_i(q_i) \leqslant \mathrm{tr}_i(p_i)$ such that $q = \overline{(q_i)}$ and $p = \overline{(p_i)}$. For any $\varepsilon > 0$, we find a unitary u_i in A_i such that (ii) of the Proposition holds. So $U = (u_i) + \mathcal{M}(\mathcal{F}, A_i)$ is a unitary demonstrating (ii) for p and q. □

References

[BH] B. Blackadar and D. Handelman, *Dimension functions and traces on C*-algebras*, J. Funct. Anal. **45** (1982), 297–340.

[D] J. Dixmier, *Les C*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1969.

[G] K. R. Goodearl, *Ring theory: Nonsingular rings and modules*, Dekker, New York, 1977.

[G2] ______, *Von Neumann regular rings*, Pitman, New York, 1979.

[Gr] F. Greenleaf, *Invariant means on topological groups*, Benjamin, New York.

[GH] K. R. Goodearl and D. E. Handelman, *Simple self-injective rings*, Comm. Algebra **3** (1975), 797–834.

[J] V. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[K] I. Kaplansky, *Rings of operators*, Benjamin, New York, 1968.

[PP] M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Sci. École Norm. Sup. (4) **19** (1986), 57–106.

[T] M. Takesaki, *Theory of operator algebras*. I, Springer-Verlag, Berlin and New York, 1979.

[W] H. Wenzl, *Representation of Hecke Algebras and subfactors*, Thesis, University of Pennsylvania, 1985.

Department of Mathematics, University of Ottawa, Ottawa, Ontario K1N 6N5, Canada

Department of Mathematics, University of California, Berkeley, California 94720