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NONCOMMUTING UNITARY GROUPS AND LOCAL BOUNDEDNESS

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ABSTRACT. We exhibit two unitary strongly continuous one-parameter groups $(e^{A_1 t})_{t\in\mathbf{R}}$ and $(e^{A_2 t})_{t\in\mathbf{R}}$ acting in a Hilbert space H, a dense subspace D of H contained in the domains of A_1 and A_2 such that $(A_1(D)\cup A_2(D))\subset D$ and $(e^{A_1 t}(D)\cup e^{A_2 t}(D))\subset D$ for each $t\in\mathbf{R}$, and an element x of D such that the function $t\to\|A_1 e^{A_2 t}x\|$ is not locally bounded.

1. Introduction. Let H be a Hilbert space. Suppose that A_1 and A_2 are the generators of unitary strongly continuous one-parameter groups $(e^{A_1 t})_{t\in\mathbf{R}}$ and $(e^{A_2 t})_{t\in\mathbf{R}}$, respectively, acting in H. Suppose, moreover, that D is a dense subspace of H contained in the domains of A_1 and A_2 such that $(A_1(D)\cup A_2(D))\subset D$ and $(e^{A_1 t}(D)\cup e^{A_2 t}(D))\subset D$ for each $t\in\mathbf{R}$. The following is an open problem: Does $A_1A_2x=A_2A_1x$ for all $x\in D$ imply that the groups $(e^{A_1 t})_{t\in\mathbf{R}}$ and $(e^{A_2 t})_{t\in\mathbf{R}}$ commute? This is a particular case of a more general question about integrability of Lie algebra representations. Under the additional assumption:

$(*)$ for each $x\in D$, the function $t\to\|A_2e^{A_2 t}x\|$ is locally bounded,

the problem above is solved in the affirmative. More generally, known results on integration of Lie algebra representations involve in a crucial way conditions analogous to $(*)$ (cf. [**1**, Theorems 3.4 and 9.1]). In this connection, P. E. T. Jørgensen and R. T. Moore (cf. [**1**, Remark, p. 67]) raised the question whether for A_1, A_2, D as above, condition $(*)$ is automatically fulfilled. We show that in general this is not the case. Our result shows, among other things, that the topological assumption on the domain D in Proposition 3.6 of [**1**] is essential.

2. The result. Given a subset A of $\mathbf{R}$ and $x\in\mathbf{R}$, let $A+x=\{y\in\mathbf{R}\colon y=a+x,\ a\in A\}$.

For each integer $n\geqslant 2$, let $I_n=(2^n,2^n+3^{-n})$ and $J_n=(2^n+2\cdot 3^{-n},\ 2^n+3^{-n+1})$.

PROPOSITION 2.1. *Given $x,y\in\mathbf{R}$, there exists at most one pair of integers $m\geqslant 2$, $n\geqslant 2$ such that*

$$(I_n+x)\cap(J_m+y)\neq\varnothing.$$

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PROOF. Suppose that for $x, y \in \mathbf{R}$ and integers $m_i \geqslant 2$, $n_i \geqslant 2$ $(i = 1, 2)$, the sets $(I_{n_i} + x) \cap (J_{m_i} + y)$ are not empty. Then

$$y - x = 2^{m_i} - 2^{n_i} + \varepsilon_i - \eta_i,$$

where $2 \cdot 3^{-m_i} < \varepsilon_i < 3^{-m_i+1}$ and $0 < \eta_i < 3^{-n_i}$. Since $|\varepsilon_1 - \eta_1| + |\varepsilon_2 - \eta_2| < 4/9 < 1$, it follows that

$$2^{m_1} - 2^{n_1} = 2^{m_2} - 2^{n_2} \tag{2.1}$$

and

$$\varepsilon_1 - \eta_1 = \varepsilon_2 - \eta_2. \tag{2.2}$$

Equation (2.1) implies that either $m_1 = m_2$ and $n_1 = n_2$, or $m_1 = n_1$ and $m_2 = n_2$. In the latter case, we have $3^{-m_i} < \varepsilon_i - \eta_i < 3^{-m_i+1}$, whence, by (2.2), $m_1 = m_2 = n_1 = n_2$. Thus, in both cases, $m_1 = m_2$ and $n_1 = n_2$.

The proof is complete.

Let $I = \bigcup_{n=2}^{\infty} I_n$ and $J = \bigcup_{n=2}^{\infty} J_n$. As an immediate corollary, we obtain

PROPOSITION 2.2. *For each $x, y \in \mathbf{R}$, the set $(I + x) \cap (J + y)$ is bounded.*

For each $n \in \mathbf{N}$, let $C^n(\mathbf{R})$ be the space of all complex functions on $\mathbf{R}$ which possess continuous derivatives of order $\leqslant n$. Let $C^\infty(\mathbf{R}) = \bigcap_{n=1}^{\infty} C^n(\mathbf{R})$. For $n \in \mathbf{N} \cup \{\infty\}$, let $C_0^n(\mathbf{R})$ be the space of functions in $C^n(\mathbf{R})$ with compact support. For any function f on $\mathbf{R}$, we denote by $\operatorname{supp} f$ the support of f.

For each integer $n \geqslant 2$, let φ_n be a nonnegative function in $C_0^\infty(\mathbf{R})$ with support in I_n, such that $|\varphi_n^{(k)}| \leqslant 1$ for $k \leqslant n$. Set

$$\varphi(x) = \sum_{n=2}^{\infty} \varphi_n(x) \qquad (x \in \mathbf{R}).$$

Clearly, $\operatorname{supp} \varphi \subset I$. Moreover, all derivatives of φ are square integrable, since for any $k \in \mathbf{N}$,

$$\int_{\mathbf{R}} \left(\varphi^{(k)}(x)\right)^2 dx \leqslant \sum_{n=2}^{k} \int_{\mathbf{R}} \left(\varphi_n^{(k)}(x)\right)^2 dx + \sum_{n=k+1}^{\infty} 3^{-n}.$$

Given an integer $n \geqslant 2$ and $x \in \mathbf{R}$, set

$$\psi_n(x) = n\delta_n^{-1}\varphi_n(x - 2 \cdot 3^{-n}),$$

where $\delta_n = \left(\int_{\mathbf{R}} \varphi_n^4(x)\, dx\right)^{1/2}$. Put $\psi(x) = \sum_{n=2}^{\infty} \psi_n(x)$ $(x \in \mathbf{R})$. Clearly, $\operatorname{supp} \psi \subset J$.

Let $L^2(\mathbf{R})$ be the Hilbert space of all (classes of) complex square integrable functions on $\mathbf{R}$, endowed with the norm $\|\cdot\|_2$.

Let π_ψ be the multiplication operator defined by $\pi_\psi f = \psi f$ $(f \in D(\pi_\psi))$, with the domain $D(\pi_\psi) = \{f \in L^2(\mathbf{R}) \colon \psi f \in L^2(\mathbf{R})\}$. π_ψ is a selfadjoint operator in $L^2(\mathbf{R})$ and $A_1 = i\pi_\psi$ is the generator of the unitary strongly continuous group $(e^{A_1 t})_{t \in \mathbf{R}}$ in $L^2(\mathbf{R})$ defined by

$$\left(e^{A_1 t} f\right)(x) = e^{it\psi(x)} f(x) \qquad \left(f \in L^2(\mathbf{R}),\ x, t \in \mathbf{R}\right).$$

Let A_2 be the generator of the unitary strongly continuous group $(e^{A_2 t})_{t \in \mathbf{R}}$ in $L^2(\mathbf{R})$ given by

$$\left(e^{A_2 t} f\right)(x) = f(x - t) \qquad \left(f \in L^2(\mathbf{R}),\ x, t \in \mathbf{R}\right).$$

It is easily checked that for any $f \in C^1(\mathbf{R}) \cap L^2(\mathbf{R})$ with $f' \in L^2(\mathbf{R})$, f is in $D(A_2)$ and $A_2 f = -f'$.

Let D_0 be the set of all functions of the form

$$x \to \varphi^{(k)}(x-u)\prod_{j=1}^{m} e^{it_j\psi(x-s_j)} \qquad \left(t_j, s_j, u \in \mathbf{R},\ m, k \in \mathbf{N}\right).$$

PROPOSITION 2.3. *The following conditions are satisfied*:

(1) $(e^{A_1 t}(D_0) \cup e^{A_2 t}(D_0)) \subset D_0$ *for each* $t \in \mathbf{R}$;

(2) $D_0 \subset D(A_1)$ *and* $A_1(D_0) \subset C_0^\infty(\mathbf{R})$;

(3) $D_0 \subset D(A_2)$ *and, for each* $f \in D_0$, $A_2 f = g_1 + g_2$ *with* $g_1 \in D_0$ *and* $g_2 \in C_0^\infty(\mathbf{R})$.

PROOF. (1) is evident.

(2) For any $f\colon\ x \to \varphi^{(k)}(x-u)\prod_{j=1}^{m} e^{it_j\psi(x-s_j)}$ $(t_j, s_j, u \in \mathbf{R},\ m, k \in \mathbf{N})$, the support of ψf is contained in $(I+u) \cap J$ and hence, by Proposition 2.2, is compact. Thus $f \in D(A_1)$ and $A_1 f \in C_0^\infty(\mathbf{R})$.

(3) If $f \in D_0$ takes the form as in the paragraph above, then, since all derivatives of φ are square integrable, f is in $D(A_2)$ and $A_2 f = g_1 + g_2$, where

$$g_1(x) = -\varphi^{(k+1)}(x-u)\prod_{j=1}^{m} e^{it_j\psi(x-s_j)} \quad \text{and} \quad g_2(x) = -if(x)\sum_{j=1}^{m} t_j\psi'(x-s_j) \qquad (x \in \mathbf{R}).$$

It is clear that $g_1 \in D_0$. Since $\operatorname{supp} g_2 \subset (I+u) \cap \bigcup_{j=1}^{m}(J+s_j)$, it follows from Proposition 2.2 that $g_2 \in C_0^\infty(\mathbf{R})$.

The proof is complete.

Let $D = \{f \in C^\infty(\mathbf{R})\colon f = f_1 + f_2,\ f_1 \in C_0^\infty(\mathbf{R}),\ f_2 \in \operatorname{span} D_0\}$, where $\operatorname{span} D_0$ denotes the linear space spanned by D_0.

We now state our major result.

THEOREM 2.4. *D is a dense subspace of $L^2(\mathbf{R})$ contained in $D(A_1) \cap D(A_2)$ such that $(A_1(D) \cup A_2(D)) \subset D$ and $(e^{A_1 t}(D) \cup e^{A_2 t}(D)) \subset D$ for each $t \in \mathbf{R}$. Moreover, φ is an element of D such that the function $t \to \|A_1 e^{A_2 t}\varphi\|_2$ is bounded in no neighborhood of* 0.

PROOF. In view of Proposition 2.3, only the last assertion requires a proof.

For each $n \in \mathbf{N}$ and each $x \in \mathbf{R}$, we have

$$\left(A_1 e^{2\cdot 3^{-n}A_2}\varphi\right)(x) = i\sum_{k,l=2}^{\infty}\psi_k(x)\varphi_l(x - 2\cdot 3^{-n}).$$

Since $\psi_n(x)\varphi_n(x - 2\cdot 3^{-n}) = n\delta_n^{-1}\varphi_n^2(x - 2\cdot 3^{-n}) \neq 0$ for some $x \in \mathbf{R}$, it follows from Proposition 2.1 that if $k \neq n$ or $l \neq n$, then $\psi_l(x)\varphi_k(x - 2\cdot 3^{-n}) = 0$ for all $x \in \mathbf{R}$. Consequently,

$$\left\|A_1 e^{2\cdot 3^{-n}A_2}\varphi\right\|_2 = n\delta_n^{-1}\left(\int_{\mathbf{R}}\varphi_n^4(x - 2\cdot 3^{-n})\,dx\right)^{1/2},$$

from which the conclusion follows.

REFERENCES

1. P. E. T. Jørgensen and R. T. Moore, *Operator commutation relations, commutation relations for operators, semigroups, and resolvents with applications to mathematical physics and representations of Lie groups*, Reidel, Dordrecht, 1984.

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