NONCOMMUTING UNITARY GROUPS
AND LOCAL BOUNDEDNESS

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Abstract. We exhibit two unitary strongly continuous one-parameter groups
\((e^{t A_1}), t \in \mathbb{R}\) and \((e^{t A_2}), t \in \mathbb{R}\) acting in a Hilbert space \(H\), a dense subspace \(D\) of \(H\)
contained in the domains of \(A_1\) and \(A_2\) such that \((A_1(D) \cup A_2(D)) \subseteq D\) and
\((e^{t A_1(D)} \cup e^{t A_2(D)}) \subseteq D\) for each \(t \in \mathbb{R}\), and an element \(x\) of \(D\) such that the
function \(t \mapsto \|A_1 e^{t A_2} x\|\) is not locally bounded.

1. Introduction. Let \(H\) be a Hilbert space. Suppose that \(A_1\) and \(A_2\) are the
generators of unitary strongly continuous one-parameter groups \((e^{t A_1}), t \in \mathbb{R}\) and
\((e^{t A_2}), t \in \mathbb{R}\), respectively, acting in \(H\). Suppose, moreover, that \(D\) is a dense subspace
of \(H\) contained in the domains of \(A_1\) and \(A_2\) such that \((A_1(D) \cup A_2(D)) \subseteq D\) and
\((e^{t A_1(D)} \cup e^{t A_2(D)}) \subseteq D\) for each \(t \in \mathbb{R}\). The following is an open problem: Does
\(A_1 A_2 x = A_2 A_1 x\) for all \(x \in D\) imply that the groups \((e^{t A_1}), t \in \mathbb{R}\) and \((e^{t A_2}), t \in \mathbb{R}\)
commute? This is a particular case of a more general question about integrability of
Lie algebra representations. Under the additional assumption:

\((*)\) \quad for each \(x \in D\), the function \(t \mapsto \|A_2 e^{t A_1} x\|\) is locally
bounded,

the problem above is solved in the affirmative. More generally, known results on
integration of Lie algebra representations involve in a crucial way conditions
analogous to \((*)\) (cf. [1, Theorems 3.4 and 9.1]). In this connection, P. E. T.
Jørgensen and R. T. Moore (cf. [1, Remark, p. 67]) raised the question whether for
\(A_1\), \(A_2\), \(D\) as above, condition \((*)\) is automatically fulfilled. We show that in general
this is not the case. Our result shows, among other things, that the topological
assumption on the domain \(D\) in Proposition 3.6 of [1] is essential.

2. The result. Given a subset \(A\) of \(\mathbb{R}\) and \(x \in \mathbb{R}\), let \(A + x = \{y \in \mathbb{R}: y = a + x, \ a \in A\}\).

For each integer \(n \geq 2\), let \(I_n = (2^n, 2^n + 3^{-n})\) and \(J_n = (2^n + 2 \cdot 3^{-n}, 2^n + 3^{-n+1})\).

Proposition 2.1. Given \(x, y \in \mathbb{R}\), there exists at most one pair of integers \(m \geq 2, n \geq 2\) such that
\[(I_n + x) \cap (J_m + y) \neq \emptyset .\]
Proof. Suppose that for \( x, y \in \mathbb{R} \) and integers \( m_i \geq 2, n_i \geq 2 \) \( (i = 1, 2) \), the sets \((I_{m_i} + x) \cap (J_{n_i} + y)\) are not empty. Then
\[
y - x = 2^{m_1} - 2^{n_1} + \varepsilon_1 - \eta_1,\]
where \( 2 \cdot 3^{-m_1} < \varepsilon_1 < 3^{-m_1+1} \) and \( 0 < \eta_1 < 3^{-n_1} \). Since \(|\varepsilon_1 - \eta_1| + |\varepsilon_2 - \eta_2| < 4/9 < 1\), it follows that
\[
(2.1) \quad 2^{m_1} - 2^{n_1} = 2^{m_2} - 2^{n_2}
\]
and
\[
(2.2) \quad \varepsilon_1 - \eta_1 = \varepsilon_2 - \eta_2.
\]
Equation (2.1) implies that either \( m_1 = m_2 \) and \( n_1 = n_2 \), or \( m_1 = n_1 \) and \( m_2 = n_2 \). In the latter case, we have \( 3^{-m_1} < \varepsilon_1 - \eta_1 < 3^{-m_1+1} \), whence, by (2.2), \( m_1 = m_2 = n_1 = n_2 \). Thus, in both cases, \( m_1 = m_2 \) and \( n_1 = n_2 \).

The proof is complete.

Let \( I = \bigcup_{n=2}^{\infty} I_n \) and \( J = \bigcup_{n=2}^{\infty} J_n \). As an immediate corollary, we obtain

**Proposition 2.2.** For each \( x, y \in \mathbb{R} \), the set \((I + x) \cap (J + y)\) is bounded.

For each \( n \in \mathbb{N} \), let \( C^n(\mathbb{R}) \) be the space of all complex functions on \( \mathbb{R} \) which possess continuous derivatives of order \( \leq n \). Let \( C^\infty(\mathbb{R}) = \bigcap_{n=1}^{\infty} C^n(\mathbb{R}) \). For \( n \in \mathbb{N} \cup \{ \infty \} \), let \( C_0^n(\mathbb{R}) \) be the space of functions in \( C^n(\mathbb{R}) \) with compact support. For any function \( f \) on \( \mathbb{R} \), we denote by \( \text{supp} f \) the support of \( f \).

For each integer \( n \geq 2 \), let \( \varphi_n \) be a nonnegative function in \( C_0^\infty(\mathbb{R}) \) with support in \( I_n \), such that \( |\varphi_n^{(k)}| \leq 1 \) for \( k \leq n \). Set
\[
\varphi(x) = \sum_{n=2}^{\infty} \varphi_n(x) \quad (x \in \mathbb{R}).
\]
Clearly, \( \text{supp} \varphi \subset I \). Moreover, all derivatives of \( \varphi \) are square integrable, since for any \( k \in \mathbb{N} \),
\[
\int_{\mathbb{R}} (\varphi^{(k)}(x))^2 \, dx \leq \sum_{n=2}^{k} \int_{\mathbb{R}} (\varphi_n^{(k)}(x))^2 \, dx + \sum_{n=k+1}^{\infty} 3^{-n}.
\]
Given an integer \( n \geq 2 \) and \( x \in \mathbb{R} \), set
\[
\psi_n(x) = n \delta_n^{-1} \varphi_n (x - 2 \cdot 3^{-n}),
\]
where \( \delta_n = (\int_{\mathbb{R}} \varphi_n^2(x) \, dx)^{1/2} \). Put \( \psi(x) = \sum_{n=2}^{\infty} \psi_n(x) (x \in \mathbb{R}) \). Clearly, \( \text{supp} \psi \subset J \).

Let \( L^2(\mathbb{R}) \) be the Hilbert space of all (classes of) complex square integrable functions on \( \mathbb{R} \), endowed with the norm \( || \cdot ||^2 \).

Let \( \pi_\psi \) be the multiplication operator defined by \( \pi_\psi f = \psi f \) \( (f \in D(\pi_\psi)) \), with the domain \( D(\pi_\psi) = \{ f \in L^2(\mathbb{R}) : \psi f \in L^2(\mathbb{R}) \} \). \( \pi_\psi \) is a selfadjoint operator in \( L^2(\mathbb{R}) \) and \( A_1 = i \pi_\psi \) is the generator of the unitary strongly continuous group \( (e^{A_1 t})_{t \in \mathbb{R}} \) in \( L^2(\mathbb{R}) \) defined by
\[
(e^{A_1 t} f)(x) = e^{i t \psi(x)} f(x) \quad (f \in L^2(\mathbb{R}), \; x, t \in \mathbb{R}).
\]

Let \( A_2 \) be the generator of the unitary strongly continuous group \( (e^{A_2 t})_{t \in \mathbb{R}} \) in \( L^2(\mathbb{R}) \) given by
\[
(e^{A_2 t} f)(x) = f(x - t) \quad (f \in L^2(\mathbb{R}), \; x, t \in \mathbb{R}).
\]
It is easily checked that for any \( f \in C^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) with \( f' \in L^2(\mathbb{R}) \), \( f \) is in \( D(A_2) \) and \( A_2f = -f' \).

Let \( D_0 \) be the set of all functions of the form

\[
x \to \varphi^{(k)}(x-u) \prod_{j=1}^{m} e^{it_j \psi(x-s_j)} (t_j, s_j, u \in \mathbb{R}, m, k \in \mathbb{N}).
\]

**Proposition 2.3.** The following conditions are satisfied:

1. \( (e^{A_t}(D_0) \cup e^{A^{2t}}(D_0)) \subset D_0 \) for each \( t \in \mathbb{R} \);
2. \( D_0 \subset D(A_1) \) and \( A_1(D_0) \subset C_0^\infty(\mathbb{R}) \);
3. \( D_0 \subset D(A_2) \) and, for each \( f \in D_0 \), \( A_2f = g_1 + g_2 \) with \( g_1 \in D_0 \) and \( g_2 \in C_0^\infty(\mathbb{R}) \).

**Proof.** (1) is evident.

(2) For any \( f: x \to \varphi^{(k)}(x-u) \prod_{j=1}^{m} e^{it_j \psi(x-s_j)} (t_j, s_j, u \in \mathbb{R}, m, k \in \mathbb{N}) \), the support of \( \psi \) is contained in \( (I + u) \cap J \) and hence, by Proposition 2.2, is compact. Thus \( f \in D(A_1) \) and \( A_1f \in C_0^\infty(\mathbb{R}) \).

(3) If \( f \in D_0 \) takes the form as in the paragraph above, then, since all derivatives of \( \varphi \) are square integrable, \( f \) is in \( D(A_2) \) and \( A_2f = g_1 + g_2 \), where

\[
g_1(x) = -\varphi^{(k+1)}(x-u) \prod_{j=1}^{m} e^{it_j \psi(x-s_j)} \quad \text{and} \quad g_2(x) = -if(x) \sum_{j=1}^{m} t_j \psi'(x-s_j)
\]

\((x \in \mathbb{R})\).

It is clear that \( g_1 \in D_0 \). Since \( \text{supp } g_2 \subset (I + u) \cap \bigcup_{j=1}^{m} (J + s_j) \), it follows from Proposition 2.2 that \( g_2 \in C_0^\infty(\mathbb{R}) \).

The proof is complete.

Let \( D = \{ f \in C^\infty(\mathbb{R}): f = f_1 + f_2, f_1 \in C_0^\infty(\mathbb{R}), f_2 \in \text{span } D_0 \} \), where \( \text{span } D_0 \) denotes the linear space spanned by \( D_0 \).

We now state our major result.

**Theorem 2.4.** \( D \) is a dense subspace of \( L^2(\mathbb{R}) \) contained in \( D(A_1) \cap D(A_2) \) such that \( (A_1(D) \cup A_2(D)) \subset D \) and \( (e^{A_t}(D) \cup e^{A^{2t}}(D)) \subset D \) for each \( t \in \mathbb{R} \). Moreover, \( \varphi \) is an element of \( D \) such that the function \( t \to \|A_1e^{A_2t}\varphi\|_2 \) is bounded in no neighborhood of 0.

**Proof.** In view of Proposition 2.3, only the last assertion requires a proof.

For each \( n \in \mathbb{N} \) and each \( x \in \mathbb{R} \), we have

\[
A_1e^{2 \cdot 3^{-n} A_2} \varphi(x) = \sum_{k,l=2}^{\infty} \psi_k(x) \varphi_l(x - 2 \cdot 3^{-n}).
\]

Since \( \psi_n(x) \varphi_n(x - 2 \cdot 3^{-n}) = n \delta_n^{-1} \varphi_n(x - 2 \cdot 3^{-n}) \neq 0 \) for some \( x \in \mathbb{R} \), it follows from Proposition 2.1 that if \( k \neq n \) or \( l \neq n \), then \( \psi_l(x) \varphi_k(x - 2 \cdot 3^{-n}) = 0 \) for all \( x \in \mathbb{R} \). Consequently,

\[
\|A_1e^{2 \cdot 3^{-n} A_2} \varphi\|_2 = n \delta_n^{-1} \left( \int_{\mathbb{R}} \varphi_n(x - 2 \cdot 3^{-n}) dx \right)^{1/2},
\]

from which the conclusion follows.
REFERENCES


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