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# AN ELEMENTARY PROOF OF A THEOREM OF S. MARDEŠIĆ

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ABSTRACT. A simple proof is given of the following theorem of S. Mardešić: *Each arc is an inverse limit of some inverse system of copies of* $[0,1]$ *with monotone bonding maps.*

A continuum is a connected and compact Hausdorff space, and an arc is a continuum with exactly two noncut points. Recall that arcs coincide with linearly ordered topological spaces which are both compact and connected. Moreover, each separable arc is homeomorphic to $[0,1]$ (see for example [**2**]).

All definitions and facts on inverse systems which we use can be found (if needed) in [**1**].

The aim of this note is to give a simple proof of the following Theorem which was derived by S. Mardešić from much more general facts (for example from a theorem [**3**] that each locally connected continuum is an inverse limit of some inverse system of metrizable locally connected continua with monotone bonding maps):

THEOREM [**3**, THEOREM 4]. *If $I$ is an arc then there is an inverse system $T = (X_a, f_b^a, A)$ such that*:

(i) *$X_a$ is homeomorphic to* $[0,1]$ *for each $a \in A$,*

(ii) *$f_b^a$ is a monotone map for every $a, b \in A$, $a \geqslant b$, and*

(iii) $I = \lim \operatorname{inv} T$.

PROOF. Let $\leqslant$ denote a natural ordering of $I$. Put

$$A = \{M \subset I \colon (M, \leqslant) \text{ is order isomorphic to the set of all rational numbers (with their usual ordering)}\}$$

and order $A$ by inclusion. Note that $(A, \subset)$ is a directed set (however, the union of two members of $A$ need not belong to $A$). For each $M \in A$ let $P_M$ denote the set of all nonisolated points of $\overline{M}$. An order-theoretic argument similar to that used in the Dedekind construction of real numbers can be used to prove the following:

if $M \in A$, $p, q \in M$, $p < q$, then there is a point $r \in \overline{M} - M$ such that $p < r < q$.

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Hence, for each $M \in A$, the set $P_M$ is closed, uncountable, and has no isolated points (use monotone sequences of points from $\overline{M} - M$ to see the last fact).

Take any $M \in A$. If $U$, $V$ are distinct components of $I - P_M$, then $\overline{U} \cap \overline{V} = \varnothing$ (because $P_M$ has no isolated points). Let $G_M$ denote the decomposition of $I$ into points and subarcs $\overline{U}$ for each component $U$ of $I - P_M$. Since each decomposition of an arc into subarcs and points is upper semicontinuous, it follows that $X_M = I/G_M$ is an arc. Let $g_M\colon I \to X_M$ denote the quotient map; so $g_M$ is monotone. Observe that $X_M$ is nondegenerate (because $P_M$ contains more than two points). Since $M$ is countable, the set $\overline{M}$ is separable. It can be easily verified that a subset of a separable linearly ordered topological space is also separable (see, for example, [**1**, Problem (read: Exercise) 3.12.4(c), p. 281]), and so $P_M$ is separable. Note that $X_M = g_M(P_M)$. Therefore $X_M$ is a separable arc, i.e., $X_M$ is homeomorphic to $[0, 1]$.

Suppose that $N \in A$ and $M \subset N$. Then $P_M \subset P_N$ and therefore $G_N$ is a refinement of $G_M$. Thus there is the unique map $f_M^N\colon X_N \to X_M$ such that

$$(*) \qquad f_M^N \circ g_N = g_M.$$

Note that $f_M^N$ is continuous, monotone, and onto. Moreover, if $K \in A$ and $K \subset M$, then $f_K^N = f_K^M \circ f_M^N$. Hence $T = (X_N, f_M^N, A)$ is an inverse system of separable arcs with monotone bonding surjections.

Denote $X = \lim\operatorname{inv} T$. By $(*)$, there is an induced continuous map $g\colon I \to X$ (this means that $g_N = f_N \circ g$, where $f_N\colon X \to X_N$ denote projections for $N \in A$). Since all the maps $g_N$ are onto, $g$ is also a surjection. Now, it suffices to prove that $g$ is one-to-one.

Suppose that $x, y \in I$, $x < y$. One can use an induction to find a set $N \in A$ such that $x < p < y$ for each $p \in N$. Then $g_N(x) \neq g_N(y)$, and so $g(x) \neq g(y)$. Therefore $g$ is a homeomorphism.

## References

1. R. Engelking, *General topology*, PWN, Polish Scientific Publishers, Warsaw, 1977.
2. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961.
3. S. Mardešić, *Locally connected, ordered and chainable continua* (in Serbo-Croatian, English summary), Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **319** (1961), 147–166; Math. Reviews **26** (1963), 5545.

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