

AN ELEMENTARY PROOF OF A THEOREM OF S. MARDEŠIĆ

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ABSTRACT. A simple proof is given of the following theorem of S. Mardešić: *Each arc is an inverse limit of some inverse system of copies of $[0, 1]$ with monotone bonding maps.*

A continuum is a connected and compact Hausdorff space, and an arc is a continuum with exactly two noncut points. Recall that arcs coincide with linearly ordered topological spaces which are both compact and connected. Moreover, each separable arc is homeomorphic to $[0, 1]$ (see for example [2]).

All definitions and facts on inverse systems which we use can be found (if needed) in [1].

The aim of this note is to give a simple proof of the following Theorem which was derived by S. Mardešić from much more general facts (for example from a theorem [3] that each locally connected continuum is an inverse limit of some inverse system of metrizable locally connected continua with monotone bonding maps):

THEOREM [3, THEOREM 4]. *If I is an arc then there is an inverse system $T = (X_a, f_b^a, A)$ such that:*

- (i) X_a is homeomorphic to $[0, 1]$ for each $a \in A$,
- (ii) f_b^a is a monotone map for every $a, b \in A$, $a \geq b$, and
- (iii) $I = \lim \operatorname{inv} T$.

PROOF. Let \leq denote a natural ordering of I . Put

$A = \{ M \subset I: (M, \leq) \text{ is order isomorphic to the set of all rational numbers (with their usual ordering)} \}$

and order A by inclusion. Note that (A, \subset) is a directed set (however, the union of two members of A need not belong to A). For each $M \in A$ let P_M denote the set of all nonisolated points of \overline{M} . An order-theoretic argument similar to that used in the Dedekind construction of real numbers can be used to prove the following:

if $M \in A$, $p, q \in M$, $p < q$, then there is a point $r \in \overline{M} - M$ such that $p < r < q$.

Received by the editors July 21, 1986.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 54F05, 54F20; Secondary 54B25.

Key words and phrases. Arc, inverse system, monotone map.

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0002-9939/87 \$1.00 + \$.25 per page

Hence, for each $M \in A$, the set P_M is closed, uncountable, and has no isolated points (use monotone sequences of points from $\overline{M} - M$ to see the last fact).

Take any $M \in A$. If U, V are distinct components of $I - P_M$, then $\overline{U} \cap \overline{V} = \emptyset$ (because P_M has no isolated points). Let G_M denote the decomposition of I into points and subarcs \overline{U} for each component U of $I - P_M$. Since each decomposition of an arc into subarcs and points is upper semicontinuous, it follows that $X_M = I/G_M$ is an arc. Let $g_M: I \rightarrow X_M$ denote the quotient map; so g_M is monotone. Observe that X_M is nondegenerate (because P_M contains more than two points). Since M is countable, the set \overline{M} is separable. It can be easily verified that a subset of a separable linearly ordered topological space is also separable (see, for example, [1, Problem (read: Exercise) 3.12.4(c), p. 281]), and so P_M is separable. Note that $X_M = g_M(P_M)$. Therefore X_M is a separable arc, i.e., X_M is homeomorphic to $[0, 1]$.

Suppose that $N \in A$ and $M \subset N$. Then $P_M \subset P_N$ and therefore G_N is a refinement of G_M . Thus there is the unique map $f_M^N: X_N \rightarrow X_M$ such that

$$(*) \quad f_M^N \circ g_N = g_M.$$

Note that f_M^N is continuous, monotone, and onto. Moreover, if $K \in A$ and $K \subset M$, then $f_K^N = f_K^M \circ f_M^N$. Hence $T = (X_N, f_M^N, A)$ is an inverse system of separable arcs with monotone bonding surjections.

Denote $X = \lim \text{inv } T$. By (*), there is an induced continuous map $g: I \rightarrow X$ (this means that $g_N = f_N \circ g$, where $f_N: X \rightarrow X_N$ denote projections for $N \in A$). Since all the maps g_N are onto, g is also a surjection. Now, it suffices to prove that g is one-to-one.

Suppose that $x, y \in I$, $x < y$. One can use an induction to find a set $N \in A$ such that $x < p < y$ for each $p \in N$. Then $g_N(x) \neq g_N(y)$, and so $g(x) \neq g(y)$. Therefore g is a homeomorphism.

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