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ON GROWTH k-ORDER OF SOLUTIONS OF A COMPLEX HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

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ABSTRACT. In this paper, we give several results about growth k-order of solutions of a complex homogeneous linear differential equation with variable coefficients, provided that the coefficients are entire functions.

Denote by C the complex field. Let $f\colon C \to C$ be an entire function. If $r > 0$, we define $\exp_1 r = \exp r$, $\exp_{k+1} r = \exp(\exp_k r)$ $(k = 1, 2, \ldots)$, and $M(r) = \max\{|f(z)|\colon |z| = r\}$. For $r > 0$ large enough, we denote $\log_1 r = \log r$, $\log_{k+1} r = \log(\log_k r)$ $(k = 1, 2, \ldots)$. The following definitions can be found in [**7, 8**]. The growth k-order $\rho_k = \rho_k(f)$ of $f(z)$ is

$$\rho_k = \limsup_{r\to\infty} (\log_{k+1} M(r)/\log r).$$

If $k = 1$, then ρ_k is called the order of $f(z)$, and we shall denote it by $\rho(f)$. We define the growth index $i(f)$ of $f(z)$ as $i(f) = 0$ if $f(z)$ is polynomial, and $i(f) = \min\{k \in \{1, 2, \ldots\}\colon \rho_k(f) < \infty\}$ if $f(z)$ is transcendental, where we set $i(f) = \infty$ when $\rho_k(f) = \infty$ for all k.

Furthermore, we shall use some notations taken from Nevanlinna theory. If $g(z)$ is a meromorphic function on C, one defines

$$m(r, g) = (1/2\pi) \int_0^{2\pi} \log^+ |g(re^{it})|\, dt,$$

$$N(r, g) = \int_0^r (n(t) - n(0))/t\, dt + n(0) \log r,$$

and $T(r, g) = m(r, g) + N(r, g)$ $(r > 0)$, where $\log^+ x = \max(0, \log x)$ for $x \geq 0$ and $n(t)$ is the number of the poles of $g(z)$ lying in $|z| \leq t$, counting according to their multiplicity. $T(r, g)$ is an increasing function of r. If $f(z)$ is entire, then $T(r, f) = m(r, f)$ and

$$T(r, f) \leq \log^+ M(r) \leq (R + r/R - r) \cdot T(R, f) \qquad (0 < r < R)$$

(see [**5**, p. 174]). From these inequalities it is easily derived that

$$\rho_k(f) = \limsup_{r\to\infty} (\log_k T(r, f)/\log r).$$

Accordingly, the k-order of a meromorphic function $g(z)$ is defined as $\rho_k(g) = \limsup_{r\to\infty} (\log_k T(r, g)/\log r)$. If $k = 1$, one obtains $\rho_1 = \rho$, the order of $g(z)$.

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Let $G \subset C$ be a simply connected domain and

$$L(w) = 0 \tag{1}$$

be a complex homogeneous linear differential equation of order n, where

$$L(w) = w^{(n)} + a_1(z)w^{(n-1)} + \cdots + a_n(z)w$$

and the a_j $(j = 1, \dots, n)$ are analytic functions on G, and at least one of them is not constant. We exclude the case $a_j = \text{constant}$ $(j = 1, \dots, n)$ because it is very well known. Each solution of (1) is analytic on G. In particular, if $G = C$, then each solution is entire.

In this paper, we give several bounds for the growth of solutions of (1), assuming that the functions a_j are entire. Some particular cases can be found in [**2**, pp. 363–366; and **4**, pp. 162–169 and 340–341]. We give some sharper results when $n = 2$.

We shall define the k-grade of the singularity of (1) as

$$\gamma_k = \sup\{\rho_k(w) \colon L(w) = 0\} \qquad (k = 1, 2, \dots),$$

and the growth index of (1) as

$$\delta = \sup\{i(w) \colon L(w) = 0\},$$

if $G = C$.

The following lemma is known for the real case (see [**3**, p. 105]).

LEMMA 1. *Assume that $G = I \times J \subset C$ is an open rectangle and $|a_j(z)| < b_j$ $(j = 1, \dots, n)$ for all $z \in G$. Define*

$$M = \tfrac{1}{2} \cdot (1 + \max(2b_1 + b_2 + \cdots + b_n, 1 + b_2, 1 + b_3, \dots, 1 + b_{n-1})).$$

Let z_0 be a point in G and $f(z)$ be a solution for (1). *Let*

$$\|f(z)\| = (|f(z)|^2 + |f'(z)|^2 + \cdots + |f^{(n-1)}(z)|^2)^{1/2}$$

and $h(z) = \exp(M(|\operatorname{Re} z - \operatorname{Re} z_0| + |\operatorname{Im} z - \operatorname{Im} z_0|))$. *Then*

$$\|f(z_0)\| \cdot (h(z))^{-1} \le \|f(z)\| \le \|f(z_0)\| \cdot h(z)$$

for every $z \in G$.

PROOF. Let $z = x + iy \in G$ and $z_0 = x_0 + iy_0$. Then $x, x_0 \in I$ and $y, y_0 \in J$. We have $L(f) = 0$ and $f = u + iv$, where $u, v \colon G \to R$. The Cauchy-Riemann equations $D_1 u = D_2 v$ and $D_x u = -D_1 v$ hold. Since $L(f) = 0$, one has $f^{(n)} = -a_1 f^{(n-1)} - \cdots - a_n f$, hence

$$|f^{(n)}| \le b_1 |f^{(n-1)}| + \cdots + b_n |f| \tag{3}$$

in G. Call $g(z) = \|f(z)\|^2$. Then $g = f\bar{f} + f'\bar{f}' + \cdots + f^{(n-1)}\overline{f^{(n-1)}}$. By partial differentiation we obtain $D_1 g = \sum_{j=0}^{n-1} (\overline{f^{(j)}} \cdot D_1 f^{(j)} + f^{(j)} \cdot D_1 \overline{f^{(j)}})$. Since each $f^{(j)}$ is analytic on G, we know that $D_1 f^{(j)} = f^{(j+1)}$ and $D_1 \overline{f^{(j)}} = D_1(D_1^j u - iD_1^j v) = \overline{D_1 f^{(j)}} = \overline{f^{(j+1)}}$. Then $D_1 g = \sum_{j=0}^{n-1} (\overline{f^{(j)}} f^{(j+1)} + f^{(j)} \overline{f^{(j+1)}})$. Thus

$$|D_1 g| \le 2 \cdot \sum_{j=n}^{n-1} |f^{(j)}| \cdot |f^{(j+1)}|. \tag{4}$$

By (3),

$$\begin{aligned}|D_1 g| &\leq 2\cdot\sum_{j=0}^{n-2}|f^{(j)}|\cdot|f^{(j+1)}|+2|f^{(n-1)}|\cdot\sum_{j=0}^{n-1}b_{n-j}|f^{(j)}|\\ &\leq \sum_{j=0}^{n-2}(|f^{(j)}|^2+|f^{(j+1)}|^2)+\sum_{j=0}^{n-1}(b_{n-j}|f^{(n-1)}|^2+|f^{(j)}|^2)\\ &=(1+b_n)|f|^2+(2+b_{n-1})|f'|^2+(2+b_{n-2})\cdot|f''|^2\\ &\quad+\cdots+(2+b_2)|f^{(n-2)}|^2+(1+2b_1+b_2+\cdots+b_n)|f^{(n-1)}|^2\\ &\leq 2M(|f|^2+\cdots+|f^{(n-1)}|^2),\end{aligned}$$

and so $|D_1 g| \leq 2Mg$, or $-2Mg \leq D_1 g \leq 2Mg$. From the right inequality, we deduce

$$\exp(-2Mx)\cdot(D_1 g(x,y)-2Mg(x,y)) = D_1(\exp(-2Mx)\cdot g(x,y)) \leq 0.$$

Fix $y\in J$. If $x > x_0$, we integrate from x_0 to x obtaining $\exp(-2Mx)g(x,y) - \exp(-2Mx_0)g(x_0,y) \leq 0$, whence $g(x,y) \leq g(x_0,y)\cdot\exp(2M(x-x_0))$. If $x < x_0$, we integrate from x to x_0 to get $\exp(-2M(x_0-x))\cdot g(x_0,y) \leq g(x,y)$. If we now use the left inequality $-2Mg \leq D_1 g$ in the same way we get $g(x,y) \leq \exp(2M(x_0-x)\cdot g(x_0,y)$ for $x < x_0$ and $\exp(-2M(x-x_0))\cdot g(x_0,y) \leq g(x,y)$ for $x_0 < x$. We can now write this as

$$\exp(-2M|x-x_0|)\cdot g(x_0,y) \leq g(x,y) \leq \exp(2M|x-x_0|)\cdot g(x_0,y). \tag{5}$$

By partial differentiation with respect to y we obtain

$$D_2 g = \sum_{j=0}^{n-1}(\overline{f^{(j)}}\cdot D_2 f^{(j)} + f^{(j)}\cdot D_2\overline{f^{(j)}}).$$

From the Cauchy-Riemann equations,

$$\begin{aligned}D_2 f^{(j)} &= D_2(D_1^j u + iD_1^j v) = -D_1^{j+1}v + iD_1^{j+1}u\\ &= i(D_1^{j+1}u + iD_1^{j+1}v) = if^{(j+1)}\end{aligned}$$

and $D_2\overline{f^{(j)}} = -i\overline{f^{(j+1)}}$. Then (4) works equally well for $D_2 g$. Consequently, $|D_2 g| \leq 2Mg$, i.e., $-2Mg(x,y) \leq D_2 g(x,y) \leq 2Mg(x,y)$. In particular

$$-2Mg(x_0,y) \leq D_2 g(x_0,y) \leq 2Mg(x_0,y)$$

and by repeating the derivation of (5) above in y instead of x we obtain

$$\exp(-2M|y-y_0|)\cdot g(x_0,y_0) \leq g(x_0,y) \leq \exp(2M|y-y_0|)\cdot g(x_0,y_0).$$

Combining with (5), one derives that

$$\begin{aligned}\exp(-2M(|x-x_0|+|y-y_0|))\cdot g(x_0,y_0) &\leq g(x,y)\\ &\leq \exp(2M(|x-x_0|+|y-y_0|))\cdot g(x_0,y_0)\end{aligned}$$

or

$$\|f(z_0)\|\cdot(h(z))^{-1} \leq \|f(z)\| \leq \|f(z_0)\|\cdot h(z). \quad \square$$

LEMMA 2. *Let $f(z)$ be a nonconstant meromorphic function on the plane. Then*

$$m(r, f'/f) = O(\log(rT(r, f))) \tag{6}$$

as $r \to \infty$ outside an exceptional set of finite linear measure. If $f(z)$ has finite order ρ, then (6) holds as $r \to \infty$ through all positive real values; if c is any number $c > 1 + 3\rho$ and $r = |z|$, then

$$|f'(z)/f(z)| < 18r^c \tag{7}$$

for r outside of intervals of finite total length.

The first part can be found in [**6**] and the second part is proved in [**4**, pp. 121–123].

LEMMA 3. *Let E be a subset of $(0, \infty)$ with finite linear measure. If $F, G\colon (0, \infty) \to R$ are functions satisfying*

(a) *F is nondecreasing and G is positive,*

(b) *$\lim_{r\to\infty} G(s(r))/G(r) = 1$ for all functions $s\colon (0, \infty) \to R$ such that $0 < s(r) - r < 1$ $(r > 0)$, then*

$$\limsup_{r\to\infty} F(r)/G(r) = \sup\left\{\limsup_{n\to\infty} F(r_n)/G(r_n)\colon r_1 < r_2 < \cdots < r_n < \cdots, \right.$$
$$\left. r_n \to \infty \text{ and } r_n \notin E \ (n = 1, 2, \dots)\right\}. \tag{8}$$

PROOF. We have

$$\limsup_{r\to\infty} F(r)/G(r) = \sup\left\{\limsup_{n\to\infty} F(r_n)/G(r_n)\colon \{r_n\}_{n=1}^{\infty} \in D\right\},$$

where $D = \{\{r_n\}_{n=1}^{\infty}\colon r_1 < r_2 < \cdots \text{ and } r_n \to \infty\}$. Take $\{r_n\}_{n=1}^{\infty} \in D$ such that $\limsup_{n\to\infty} F(r_n)/G(r_n) = \limsup_{r\to\infty} F(r)/G(r)$. One can find a sequence $\{s_n\}_{n=1}^{\infty}$ and an integer n_0 with $s_1 < s_2 < \cdots < s_n < \cdots$, $s_n \notin E$ for every n, and $0 < s_n - r_n < 1$ $(n > n_0)$, because E has finite measure. Then $\{s_n\}_{n=1}^{\infty} \in D$ and $\lim_{n\to\infty} G(s_n)/G(r_n) = 1$, so

$$\begin{aligned}\limsup_{n\to\infty} F(r_n)/G(r_n) &= \limsup_{n\to\infty} F(r_n)/G(s_n)\\ &\le \limsup_{n\to\infty} F(s_n)/G(s_n) \le \limsup_{r\to\infty} F(r)/G(r),\end{aligned}$$

and (8) follows. □

Now, we state our result on growth. Note that part (iv) generalizes a theorem of Wittich (1948): If all solutions are of finite order, then the coefficients are polynomials. For this, see [**4**, pp. 167–168].

THEOREM 4. *Assume that $G = C$ in (1). Let $p = \max\{i(a_j)\colon j = 1, \dots, n\}$. If p is finite and $p > 0$, then we call $\alpha = \max\{\rho_p(a_j)\colon j = 1, \dots, n\}$. Then the following conditions are satisfied:*

(i) *$\delta \le 1 + p$.*

(ii) *If p is finite and $p > 0$, then $\gamma_{p+1} \le \alpha$.*

(iii) *If every $a_j(z)$ is a polynomial, let $m = \max\{\text{degree}(a_j)\colon j = 1, \dots, n\}$. Then*

$$\gamma_1 \le 1 + m. \tag{9}$$

(iv) *Let* $n = 2$. *Then* $\delta = 1 + p$. *In addition,* $\gamma_{p+1} = \alpha$ *if* $p > 0$, *and* $\gamma_1 \geq \frac{1}{3}(\text{degree}(a_1) - 1)$ *if* $p = 0$.

PROOF. Let $f(z)$ be a solution for (1). Call $M(r) = \max\{|f(z)|: |z| = r\}$, $M_j(r) = \max\{|a_j(z)|: |z| = r\}$, $N(r) = \sup\{|f(z)|: z \in G_r\}$ and $N_j(r) = \sup\{|a_j(z)|: z \in G_r\}$ $(j = 1, \ldots, n), G_r$ being $G_r = (-r, r) \times (-r, r) \subset C$. From the inequalities $N(r/2) \leq M(r) \leq N(2r)$ and $N_j(r/2) \leq M_j(r) \leq N_j(2r)$, it is obvious that one can replace $M(r)$ by $N(r)$ and $M_j(r)$ by $N_j(r)$ in the respective expressions of k-order.

If $p = \infty$, then (i) is trivial. Otherwise, there exist real numbers $r_0, d_1, \ldots, d_n > 0$ such that $N_j(r) < \exp_p(r^{d_j})$ $(j = 1, \ldots, n; r > r_0)$. Apply Lemma 1 for $z_0 = 0$ and G_r. We obtain $|f(z)| \leq \|f(z)\| \leq \|f(0)\| \cdot \exp(2r + 2rN_1(r) + \cdots + 2rN_n(r))$ if $z \in G_r$ and $r > r_0$, because $M \leq 1 + N_1(r) + \cdots + N_n(r)$. If $d \in (1 + \max(d_1, \ldots, d_n), \infty)$, one can find $r_1 > r_0$ with $2r + \cdots + 2rN_n(r) < \exp_p(r^d)$ when $r > r_1$. If $s \in (d, \infty)$, there is an $r_2 > r_1$ such that $\|f(0)\| \cdot \exp(2r + \cdots + 2rN_n(r)) < \exp_{p+1}(r^s)$ when $r > r_2$. Thus $N(r) < \exp_{p+1}(r^s)$ and $(\log_{p+2} N(r))/\log r < s$ asymptotically (asymp.). Hence $\rho_{p+1}(f)$ is finite and $i(f) \leq 1 + p$, so $\delta \leq 1 + p$.

(ii) From the definition of α, it is clear that, given $\varepsilon > 0$, one has $N_j(r) < \exp_p(r^{\alpha+\varepsilon})$ for $r > r_0$. But it is easily derived that there is $r_1 > r_0$ with $\|f(0)\| \cdot \exp(2r + \cdots + 2rN_n(r)) < \exp_{p+1}(r^{\alpha+2\varepsilon})$, whence $N(r) < \exp_{p+1}(r^{\alpha+2\varepsilon})$ asymp. Hence, $(\log_{p+2} N(r))/\log r < \alpha + 2\varepsilon$ asymp. and $\rho_{p+1}(f) \leq \alpha + 2\varepsilon$ for each $\varepsilon > 0$, i.e., $\rho_{p+1}(f) \leq \alpha$ and $\delta_{p+1} \leq \alpha$.

Similar arguments to those of (ii) let us prove (iii), taking into account that $2r + 2rN_1(r) + \cdots + 2rN_n(r) < r^{m+1+\varepsilon}$ for all $r > r_0(\varepsilon) > 0$. Hence, (9) holds.

(iv) In this case, $L(w) = w'' + a_1(z)w' + a_2(z)w$. Let $\{w_1, w_2\}$ be a fundamental system of solutions. The wronskian $W = w_1 w_2' - w_1' w_2$ is entire and $\rho_k(W) \leq \max(\rho_k(w_1), \rho_k(w_2))$, because $\max(\rho_k(gh), \rho_k(g+h)) \leq \max(\rho_k(g), \rho_k(h))$ and $\rho_k(g') = \rho_k(g)$ if g, h are entire (see, for instance, [**1**, pp. 28, 37, and 43]).

Then $a_1(z) = -W'/W$ and $a_2(z) = -(w''/w) - a_1(w'/w) = -(w''/w') \cdot (w'/w) - a_1(w'/w)$ if $L(w) = 0$. Furthermore, $\gamma_k = \max(\rho_k(w_1), \rho_k(w_2))$, because $w = aw_1 + bw_2$ if $L(w) = 0$ $(a, b \in C$, depending on $w)$. From Nevanlinna theory, $m(r, a_2) = m(r, w''/w') + 2m(r, w'/w) + m(r, a_1) + O(\log r)$.

If $\delta = 0$, then $i(w) = 0$ for all solutions w, that is, every solution is a polynomial. But it is well known that this is possible only if $L(w) = w''$ and this case is excluded. Then $\delta = 1 + p$ is trivial if $p = 0$.

Let $p > 0$. By contradiction, assume $0 < \delta < 1 + p$. then $\delta \leq p$ and $i(w) \leq p$ if $L(w) = 0$. Hence $\rho_p(w_j)$ is finite $(j = 1, 2)$ so $\rho_p(W)$ is finite. Then $m(r, a_1) = m(r, W'/W) = O(\log(r \cdot T(r, W)))$ outside E with finite measure by Lemma 2. We have

$$\log_{p-1} m(r, a_1) = O(\log_p(r \cdot T(r, W))) = O(\log_p r + \log_p T(r, W)) = O(\log r)$$

outside E, because $\log_p T(r, W) = O(\log r)$. By using Lemma 3, with $F(r) = \log_{p-1} m(r, a_1)$ and $G(r) = \log r$, we conclude that $\rho_{p-1}(a_1)$ is finite if $p > 1$, and a_1 is a polynomial if $p = 1$. But, again from Lemma 2, outside a set E as above,

$$m(r, a_2) = O(\log(rT(r, w))) + O(\log(rT(r, w'))) + O(\log(rT(r, a_1))) + O(\log r). \tag{10}$$

Since $\rho_p(w) = \rho_p(w')$, we reason as above that $\rho_{p-1}(a_2)$ is finite if $p > 1$ or a_2 is a polynomial if $p = 1$. Thus $p = \max(i(a_1), i(a_2)) \leq p - 1$, a contradiction. Then $\delta = 1 + p$.

If $p > 0$, assume $\gamma_{p+1} < \alpha$. Then $\alpha > 0$, $\rho_{p+1}(w) < c$ if $L(w) = 0$, and $\rho_{p+1}(W) < c$ for some $c \in (0, \alpha)$. Given $\varepsilon > 0$, $\log_{p+1} T(r, W) < (c + \varepsilon) \log r$ asymp. for every $\varepsilon > 0$. Then

$$\log_p m(r, a_1) = \log_p(O(\log(rT(r, W)))) < (c + 2\varepsilon) \log r$$

outside $E = E(\varepsilon)$ with finite measure. From Lemma 3,

$$\limsup_{r \to \infty} (\log_p m(r, a) / \log r) < c + 2\varepsilon$$

for every $\varepsilon > 0$, i.e., $\rho_p(a_1) \leq c$. From (10), one obtains similarly $\rho_p(a_2) \leq c$. Hence $\alpha = \max(\rho_p(a_1), \rho_p(a_2)) < \alpha$, a contradiction, so $\gamma_{p+1} = \alpha$.

If $p = 0$, assume $\gamma_1 < L$, where $L = \frac{1}{3}(\text{degree}(a_1) - 1)$. then $\text{degree}(a_1) > 1$, $\rho(w) < d$ if $L(w) = 0$, and $\rho(W) < d$ for some $d \in (0, L)$. Let $c > 1 + 3\rho(W)$. By Lemma 2 and (7), we have

$$m(r, a_1) = m(r, W'/W) = (1/2\pi) \cdot \int_0^{2\pi} \log^+ |W'(re^{it})/W(re^{it})| \, dt < c \cdot \log r$$

for r outside $E = E(c)$ of finite linear measure. By Lemma 3,

$$\limsup_{r \to \infty} m(r, a_1) / \log r \leq c,$$

whence $\limsup_{r \to \infty} m(r, a_1) / \log r \leq 1 + 3\rho(W)$. From Nevanlinna theory,

$$\text{degree}(a_1) \leq 1 + 3\rho(W) < 1 + 3d < 1 + 3L = \text{degree}(a_1),$$

a contradiction. Thus $\gamma_1 \geq L$. The theorem is proved. □

References

1. L. Bernal, Tesis, Facultad de Matemáticas, Universidad de Sevilla, Spain, 1984.
2. L. Bieberbach, *Theorie der gewöhnlichen differentialgleichungen*, Springer-Verlag, Berlin and New York, 1965.
3. E. A. Coddington, *An introduction to ordinary differential equations*, 5th printing, Prentice-Hall, Englewood Cliffs, N.J. 1964.
4. E. Hille, *Ordinary differential equations in the complex domain*, Wiley, New York, 1984.
5. A. S. B. Holland, *Introduction to the theory of entire functions*, Academic Press, New York and London, 1973.
6. R. Nevanlinna, *Analytic functions*, Springer-Verlag, Berlin and New York, 1970.
7. A. R. Reddy, *On entire Dirichlet series of infinite order*, Rev. Mat. Hisp.-Amer. **27** (1967), 120–131.
8. D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc. **69** (1963), 410–414.

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