ON THE DERIVATIVE 
WITH RESPECT TO A POINT

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ABSTRACT. The derivative of a polynomial $p(z)$ with respect to a point $\zeta$ is defined by the formula $A_\zeta p(z) = (\zeta - z)p'(z) + np(z)$, where $n$ is the degree of the polynomial. Let $p(z)$ have all its zeros in the unit disk and one zero at $z = 1$. We determine a minimal region that must contain at least one zero of $A_\zeta p(z)$.

1. Introduction. It is customary to define the derivative of a polynomial $p(z)$ with respect to a point $\zeta$ by the formula

$$A_\zeta \equiv A_\zeta(p(z)) \equiv (\zeta - z)p'(z) + np(z),$$

where $n$ is the degree of the polynomial (see [3, vol. 1, p. 48; 4, vol. 2, p. 61]). Clearly $\lim A_\zeta(p(z))/\zeta = p'(z)$ as $\zeta \to \infty$. Thus $p'(z)$ can be regarded as the derivative of $p(z)$ with respect to $\zeta = \infty$. Hence, theorems about $A_\zeta(p(z))$ can be regarded as generalizations for theorems about the derivative of $p(z)$.

An interesting conjecture due to Sendov runs as follows.

CONJECTURE. Let $p(z)$ be a polynomial with all of its zeros in $\mathbb{E}$: $|z| \leq 1$. If $a$ is any one of these zeros, then $p'(z)$ has at least one zero in the disk $|z - a| \leq 1$.

This conjecture is still unsettled, although it is known to be true in many special cases. For details and further references see Schmeisser [6, 7]. One can always rotate the disk, and hence one can always assume that $0 \leq a \leq 1$ without loss of generality. Our interest centers on the special case that $a = 1$, where we have

THEOREM A. Let $p(z)$ be a polynomial with all its zeros in $\mathbb{E}$ and assume that $p(1) = 0$. Then $p'(z)$ has at least one zero in $|z - 1/2| \leq 1/2$ (see Goodman, Rahman, and Ratti [2] and Schmeisser [5]).

Here we consider the generalization of Theorem A to $A_\zeta(p(z))$, the derivative of $p(z)$ with respect to the point $\zeta$. For each fixed $\zeta$ we obtain a minimal set in which $A_\zeta$ must have at least one zero.

2. Statement of the result. If $\zeta = 1$ and $p(1) = 0$, then it is clear from (1.1) that $A_\zeta(p(z)) = 0$ when $z = 1$. Thus the singleton set $\{1\}$ forms the minimal set when $\zeta = 1$. Thus the case $\zeta = 1$ is trivial. The case $\zeta = \infty$ can also be set aside because this is just the case settled by Theorem A. Henceforth we assume that $\zeta \neq 1$ and $\zeta \neq \infty$.

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THEOREM 1. Let \( p(z) \) be a polynomial with all of its zeros in \( \hat{E} \) and one zero at \( z = 1 \). If \( \zeta \neq 1 \) and \( \zeta \neq \infty \), then \( A_\zeta(p(z)) \) has at least one zero in \( \hat{K} \), where \( \hat{K} \) is the image of \( \hat{E} \) under the transformation

\[
(2.1) \quad w = L(z) \equiv \frac{(\zeta - 2)z + \zeta}{-z + (2\zeta - 1)}.
\]

This result is sharp. This means that \( \hat{K} \) is a minimal set (if any point is deleted from \( \hat{K} \), then the assertion is false).

The mapping (2.1) has several interesting properties. First, \( L(\hat{E}) = 1 \) iff \( \zeta = 1 \); a trivial case that has already been discussed.

Second, \( L(\hat{E}) \) is the half-plane \( \text{Re } w \leq 1 \) iff \( \zeta \) is on the circle \( |\zeta - 1/2| = 1/2 \) and \( \zeta \neq 1 \).

Third, \( \hat{K} \) is always symmetric with respect to the real axis, and always contains the points \( L(1) = 1 \) and \( L(-1) = 1/\zeta \).

Fourth, \( \zeta \) is a fixed point of (2.1).

Finally, \( L(\hat{E}) = \hat{E} \) if and only if \( |\zeta| = 1 \) and \( \zeta \neq 1 \).

3. Proof of Theorem 1. As in [2] we map \( \hat{E} \) onto the half-plane \( \text{Re } w \geq 0 \) by the Möbius transformation

\[
(3.1) \quad w = M(z) = \frac{1 + z}{1 - z}, \quad z = M^{-1}(w) = \frac{w - 1}{w + 1}.
\]

The equation \( A_\zeta(p(z)) = 0 \) is satisfied whenever

\[
(3.2) \quad \frac{A_\zeta(p(z))}{p(z)} = (\zeta - z) \frac{p'(z)}{p(z)} + n = 0
\]

or whenever

\[
(3.3) \quad (\zeta - z) \left( \frac{1}{z - 1} + \sum_{k=1}^{n-1} \frac{1}{z - z_k} \right) + n = 0.
\]

Here we let \( z_1, z_2, \ldots, z_{n-1} \) and \( z_n = 1 \) be the zeros of \( p(z) \). If \( w_k = M(z_k) \) for \( k = 1, 2, \ldots, n - 1 \) and \( \zeta^* = M(\zeta) \), then the transformation (3.1) applied to (3.3) gives

\[
(3.4) \quad 2 \frac{\zeta^* - w}{(\zeta^* + 1)(w + 1)} \left( \frac{w + 1}{-2} + \sum_{k=1}^{n-1} \frac{(w + 1)(w_k + 1)}{2(w - w_k)} \right) + n = 0.
\]

After a few simple steps we arrive at

\[
(3.5) \quad -1 + \sum_{k=1}^{n-1} \frac{w_k + 1}{w - w_k} + n \frac{\zeta^* + 1}{\zeta^* - w} = 0.
\]

We add 1 to each term under the sum sign, subtract \( n \) from the last term, and divide by \( w + 1 \). Here we observe that \( w = -1 = M(\infty) \). Then (3.5) gives

\[
(3.6) \quad \sum_{k=1}^{n-1} \frac{1}{w - w_k} - \frac{n}{w - \zeta^*} = 0.
\]
Consequently if $\eta_k$ is any root of equation (3.6), it is also a root of (3.4), and $Z_k = M^{-1}(\eta_k)$ is a zero of $A_\zeta(p(z))$. We next put (3.6) in the form $N(w)/D(w) = 0$, where $N$ and $D$ are polynomials, and find that

\[(3.7) \quad N(w) = w^{n-1} + [(n-1)\zeta^* - 2(w_1 + w_2 + \cdots + w_{n-1})]w^{n-2} + \cdots.\]

Therefore if $\eta_1, \eta_2, \ldots, \eta_{n-1}$ are the roots of $N(w) = 0$, then

\[(3.8) \quad \eta_1 + \eta_2 + \cdots + \eta_{n-1} = 2(w_1 + w_2 + \cdots + w_{n-1}) - (n-1)\zeta^*.\]

Since $\Re w_k \geq 0$ for $k = 1, 2, \ldots, n-1$, equation (3.8) tells us that there exists at least one $\eta^*$ which is a root of $N(w) = 0$, and for which

\[(3.9) \quad \Re \eta^* \geq \Re(-\zeta^*).\]

Then $Z_c = M^{-1}(\eta^*)$ is a root of $A_\zeta(p(z)) = 0$. Since (3.9) defines a half-plane $K^*: \Re w \geq \Re(-\zeta^*)$, the image $\bar{K}$ of this half-plane under $z = M^{-1}(w)$ must contain at least one root or $A_\zeta(p(z)) = 0$.

Let $\eta^*$ be any fixed point in $K^*$, where $\zeta$ and $\zeta^*$ are fixed. We will prove that $\eta^*$ can occur as a zero of $N(w)$. To see this set $n = 2$. Then equation (3.6) gives

\[(3.10) \quad \frac{1}{w - w_1} = \frac{2}{w - \zeta^*}\]

and hence $w = \eta^* = 2w_1 - \zeta^*$. Thus by selecting $w_1$ suitably with $\Re w_1 \geq 0$ we can force $\eta^*$ to be any preassigned point in $K^*$. Thus, the set $\bar{K} = M^{-1}(K^*)$ is a minimal set under the conditions of Theorem 1. Unless $\zeta = 1$, the set $\bar{K}$ is always a closed circular region, which may be a half-plane, or the closed exterior of a circle. Further, $\bar{K}$ is symmetric with respect to the real axis because it is the image of $\Re w \geq \Re(-\zeta^*)$ under $M(w) = (w - 1)/(w + 1)$. Since the set $K^*$ is a minimal set for a root of (3.6), it follows that $\bar{K}$ is a minimal set for zeros of $A_\zeta(p(z))$. Finally, we observe that points of $\bar{K}$ occur for the quadratic $p(z) = (z - 1)(z - z_1)$. For this polynomial we find that

\[(3.11) \quad A_\zeta(p(z)) = (2\zeta - z_1 - 1)z - (\zeta z_1 - 2z_1 + \zeta).\]

Hence $A_\zeta(p(z)) = 0$ for $z = z_c$, where

\[(3.12) \quad z_c = \frac{(\zeta - 2)z_1 + \zeta}{-z_1 + 2\zeta - 1}, \quad z_1 \neq 2\zeta - 1.\]

Thus as $z_1$ ranges over $E$, equation (3.12) or $L(z)$ generates $\bar{K}$. \(\Box\)

**4. Some concluding remarks.** Under the conditions of Theorem 1, equation (2.1) provides the complete solution to the problem of finding a minimal set for some zero of $A_\zeta(p(z))$ when $p(z)$ is any polynomial with one zero at $z = 1$ and all zeros in $E$. Suppose, however, that we admit to competition only those polynomials of degree $n$ where $n$ is a fixed integer and $n > 2$. Then a minimal set, which now depends on $n$, may be different from the one found in Theorem 1. To illustrate this possibility, let us return to the original conjecture of Sendov, modified as follows.

**Problem.** For fixed $n$, let $p(z)$ be a polynomial of degree $n$ with all of its zeros in $E$ and one zero at $z = a$ where $a$ is fixed in $[0, 1]$. Find a minimal set $S(n, a)$ that must always contain at least one zero of $p'(z)$.

It is easy to prove that when $a = 0$, one such minimal set is the disk

\[(4.1) \quad |z| \leq (1/n)^{1/(n-1)}.\]
Thus, in this case, $S(n, 0)$ is a strictly increasing sequence of regions as $n$ increases.

When $a = 1$, a minimal set $S(n, 1)$ is known to be the disk $|z - 1/2| \leq 1/2$ when $n = 2$ or when $n = 3$. The assertion is trivial when $n = 2$ and the case $n = 3$ was settled in [2]. As far as the author is aware the problem is still open for $n \geq 4$. Since Theorem A covers all polynomials, the set $S(n, 1)$ must be contained in the disk $|z - 1/2| \leq 1/2$, but may in fact be smaller than this disk.

REFERENCES


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