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ON THE DERIVATIVE WITH RESPECT TO A POINT

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ABSTRACT. The derivative of a polynomial $p(z)$ with respect to a point ς is defined by the formula $A_\varsigma p(z) = (\varsigma - z)p'(z) + np(z)$, where n is the degree of the polynomial. Let $p(z)$ have all its zeros in the unit disk and one zero at $z = 1$. We determine a minimal region that must contain at least one zero of $A_\varsigma p(z)$.

1. Introduction. It is customary to define the derivative of a polynomial $p(z)$ with respect to a point ς by the formula

$$A_\varsigma \equiv A_\varsigma(p(z)) \equiv (\varsigma - z)p'(z) + np(z), \tag{1.1}$$

where n is the degree of the polynomial (see [**3**, vol. 1, p. 48; **4**, vol. 2, p. 61]). Clearly $\lim A_\varsigma(p(z))/\varsigma \equiv p'(z)$ as $\varsigma \to \infty$. Thus $p'(z)$ can be regarded as the derivative of $p(z)$ with respect to $\varsigma = \infty$. Hence, theorems about $A_\varsigma(p(z))$ can be regarded as generalizations for theorems about the derivative of $p(z)$.

An interesting conjecture due to Sendov runs as follows.

CONJECTURE. Let $p(z)$ be a polynomial with all of its zeros in $\bar{E}$: $|z| \le 1$. If a is any one of these zeros, then $p'(z)$ has at least one zero in the disk $|z - a| \le 1$.

This conjecture is still unsettled, although it is known to be true in many special cases. For details and further references see Schmeisser [**6**, **7**]. One can always rotate the disk, and hence one can always assume that $0 \le a \le 1$ without loss of generality. Our interest centers on the special case that $a = 1$, where we have

THEOREM A. *Let $p(z)$ be a polynomial with all its zeros in $\bar{E}$ and assume that $p(1) = 0$. Then $p'(z)$ has at least one zero in $|z - 1/2| \le 1/2$ (see Goodman, Rahman, and Ratti* [**2**] *and Schmeisser* [**5**]).

Here we consider the generalization of Theorem A to $A_\varsigma(p(z))$, the derivative of $p(z)$ with respect to the point ς. For each fixed ς we obtain a minimal set in which A_ς must have at least one zero.

2. Statement of the result. If $\varsigma = 1$ and $p(1) = 0$, then it is clear from (1.1) that $A_\varsigma(p(z)) = 0$ when $z = 1$. Thus the singleton set $\{1\}$ forms the minimal set when $\varsigma = 1$. Thus the case $\varsigma = 1$ is trivial. The case $\varsigma = \infty$ can also be set aside because this is just the case settled by Theorem A. Henceforth we assume that $\varsigma \ne 1$ and $\varsigma \ne \infty$.

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THEOREM 1. *Let $p(z)$ be a polynomial with all of its zeros in $\bar{E}$ and one zero at $z = 1$. If $\varsigma \neq 1$ and $\varsigma \neq \infty$, then $A_\varsigma(p(z))$ has at least one zero in $\bar{K}$, where $\bar{K}$ is the image of $\bar{E}$ under the transformation*

$$w = L(z) \equiv \frac{(\varsigma - 2)z + \varsigma}{-z + (2\varsigma - 1)}. \tag{2.1}$$

This result is sharp. This means that $\bar{K}$ is a minimal set (if any point is deleted from $\bar{K}$, then the assertion is false).

The mapping (2.1) has several interesting properties. First, $L(\bar{E}) = 1$ iff $\varsigma = 1$; a trivial case that has already been discussed.

Second, $L(\bar{E})$ is the half-plane $\operatorname{Re} w \leq 1$ iff ς is on the circle $|\varsigma - 1/2| = 1/2$ and $\varsigma \neq 1$.

Third, $\bar{K}$ is always symmetric with respect to the real axis, and always contains the points $L(1) = 1$ and $L(-1) = 1/\varsigma$.

Fourth, ς is a fixed point of (2.1).

Finally, $L(\bar{E}) = \bar{E}$ if and only if $|\varsigma| = 1$ and $\varsigma \neq 1$.

3. Proof of Theorem 1. As in [**2**] we map $\bar{E}$ onto the half-plane $\operatorname{Re} w \geq 0$ by the Möbius transformation

$$w = M(z) = \frac{1+z}{1-z}, \qquad z = M^{-1}(w) = \frac{w-1}{w+1}. \tag{3.1}$$

The equation $A_\varsigma(p(z)) = 0$ is satisfied whenever

$$\frac{A_\varsigma(p(z))}{p(z)} = (\varsigma - z)\frac{p'(z)}{p(z)} + n = 0 \tag{3.2}$$

or whenever

$$(\varsigma - z)\left(\frac{1}{z-1} + \sum_{k=1}^{n-1} \frac{1}{z - z_k}\right) + n = 0. \tag{3.3}$$

Here we let $z_1, z_2, \ldots, z_{n-1}$ and $z_n = 1$ be the zeros of $p(z)$. If $w_k = M(z_k)$ for $k = 1, 2, \ldots, n-1$ and $\varsigma^* = M(\varsigma)$, then the transformation (3.1) applied to (3.3) gives

$$2\frac{\varsigma^* - w}{(\varsigma^* + 1)(w+1)}\left(\frac{w+1}{-2} + \sum_{k=1}^{n-1} \frac{(w+1)(w_k+1)}{2(w - w_k)}\right) + n = 0. \tag{3.4}$$

After a few simple steps we arrive at

$$-1 + \sum_{k=1}^{n-1} \frac{w_k + 1}{w - w_k} + n\frac{\varsigma^* + 1}{\varsigma^* - w} = 0. \tag{3.5}$$

We add 1 to each term under the sum sign, subtract n from the last term, and divide by $w + 1$. Here we observe that $w = -1 = M(\infty)$. Then (3.5) gives

$$\sum_{k=1}^{n-1} \frac{1}{w - w_k} - \frac{n}{w - \varsigma^*} = 0. \tag{3.6}$$

Consequently if η_k is any root of equation (3.6), it is also a root of (3.4), and $Z_k = M^{-1}(\eta_k)$ is a zero of $A_\varsigma(p(z))$. We next put (3.6) in the form $N(w)/D(w) = 0$, where N and D are polynomials, and find that

$$N(w) = w^{n-1} + [(n-1)\varsigma^* - 2(w_1 + w_2 + \cdots + w_{n-1})]w^{n-2} + \cdots . \tag{3.7}$$

Therefore if $\eta_1, \eta_2, \ldots, \eta_{n-1}$ are the roots of $N(w) = 0$, then

$$\eta_1 + \eta_2 + \cdots + \eta_{n-1} = 2(w_1 + w_2 + \cdots + w_{n-1}) - (n-1)\varsigma^*. \tag{3.8}$$

Since $\operatorname{Re} w_k \geq 0$ for $k = 1, 2, \ldots, n-1$, equation (3.8) tells us that there exists at least one η^* which is a root of $N(w) = 0$, and for which

$$\operatorname{Re} \eta^* \geq \operatorname{Re}(-\varsigma^*). \tag{3.9}$$

Then $Z_c = M^{-1}(\eta^*)$ is a root of $A_\varsigma(p(z)) = 0$. Since (3.9) defines a half-plane K^*: $\operatorname{Re} w \geq \operatorname{Re}(-\varsigma^*)$, the image $\bar{K}$ of this half-plane under $z = M^{-1}(w)$ must contain at least one root or $A_\varsigma(p(z)) = 0$.

Let η^* be any fixed point in K^*, where ς and ς^* are fixed. We will prove that η^* can occur as a zero of $N(w)$. To see this set $n = 2$. Then equation (3.6) gives

$$1/(w - w_1) = 2/(w - \varsigma^*) \tag{3.10}$$

and hence $w = \eta^* = 2w_1 - \varsigma^*$. Thus by selecting w_1 suitably with $\operatorname{Re} w_1 \geq 0$ we can force η^* to be any preassigned point in K^*. Thus, the set $\bar{K} = M^{-1}(K^*)$ is a minimal set under the conditions of Theorem 1. Unless $\varsigma = 1$, the set $\bar{K}$ is always a closed circular region, which may be a half-plane, or the closed exterior of a circle. Further, $\bar{K}$ is symmetric with respect to the real axis because it is the image of $\operatorname{Re} w \geq \operatorname{Re}(-\varsigma^*)$ under $M(w) = (w-1)/(w+1)$. Since the set K^* is a minimal set for a root of (3.6), it follows that $\bar{K}$ is a minimal set for zeros of $A_\varsigma(p(z))$. Finally, we observe that points of $\bar{K}$ occur for the quadratic $p(z) = (z-1)(z-z_1)$. For this polynomial we find that

$$A_\varsigma(p(z)) = (2\varsigma - z_1 - 1)z - (\varsigma z_1 - 2z_1 + \varsigma). \tag{3.11}$$

Hence $A_\varsigma(p(z)) = 0$ for $z = z_c$, where

$$z_c = \frac{(\varsigma - 2)z_1 + \varsigma}{-z_1 + 2\varsigma - 1}, \qquad z_1 \neq 2\varsigma - 1. \tag{3.12}$$

Thus as z_1 ranges over $\bar{E}$, equation (3.12) or $L(z)$ generates $\bar{K}$. □

4. Some concluding remarks. Under the conditions of Theorem 1, equation (2.1) provides the complete solution to the problem of finding a minimal set for some zero of $A_\varsigma(p(z))$ when $p(z)$ is any polynomial with one zero at $z = 1$ and all zeros in $\bar{E}$. Suppose, however, that we admit to competition only those polynomials of degree n where n is a fixed integer and $n > 2$. Then a minimal set, which now depends on n, may be different from the one found in Theorem 1. To illustrate this possibility, let us return to the original conjecture of Sendov, modified as follows.

PROBLEM. For fixed n, let $p(z)$ be a polynomial of degree n with all of its zeros in $\bar{E}$ and one zero at $z = a$ where a is fixed in $[0, 1]$. Find a minimal set $S(n, a)$ that must always contain at least one zero of $p'(z)$.

It is easy to prove that when $a = 0$, one such minimal set is the disk

$$|z| \leq (1/n)^{1/(n-1)}. \tag{4.1}$$

Thus, in this case, $S(n,0)$ is a strictly increasing sequence of regions as n increases.

When $a = 1$, a minimal set $S(n,1)$ is known to be the disk $|z - 1/2| \leq 1/2$ when $n = 2$ or when $n = 3$. The assertion is trivial when $n = 2$ and the case $n = 3$ was settled in [**2**]. As far as the author is aware the problem is still open for $n \geq 4$. Since Theorem A covers all polynomials, the set $S(n,1)$ must be contained in the disk $|z - 1/2| \leq 1/2$, but may in fact be smaller than this disk.

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