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SPECTRAL SYNTHESIS ON THE ALGEBRA OF ABSOLUTELY CONVERGENT LAGUERRE POLYNOMIAL SERIES

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ABSTRACT. Askey and Gasper [**1**] constructed the algebra with convolution structure for Laguerre polynomials. We will answer the question of spectral synthesis of the one point on this algebra.

1. Introduction. Let $L_n^\alpha(x)$ be the Laguerre polynomial given by

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!}\left(\frac{d}{dx}\right)^n [e^{-x}x^{n+\alpha}],$$

and denote by $R_n^\alpha(x)$ the normalized Laguerre polynomial so that

$$R_n^\alpha(x) = L_n^\alpha(x)/L_n^\alpha(0),$$

where $\alpha > -1$ and n is a nonnegative integer.

Let $\alpha \geq -1/2$ and $\tau \geq 2$ or let $\alpha \geq \alpha_0 = (-5+(17)^{1/2})/2$ and $\tau \geq 1$. Let $A^{(\alpha,\tau)}$ be the space

$$\left\{ f(x) \text{ on } [0,\infty); f(x) = \sum_{n=0}^{\infty} a_n R_n^\alpha(x) e^{-\tau x}, \sum_{n=0}^{\infty} |a_n| < \infty \right\},$$

and introduce a norm to $A^{(\alpha,\tau)}$ by $\|f\| = \sum_{n=0}^{\infty} |a_n|$. Then Askey and Gasper [**1**] showed that

(A) [**1**, §§4, 5] $A^{(\alpha,\tau)}$ is a Banach algebra of continuous functions on the interval $[0,\infty)$ vanishing at infinity with the product of pointwise multiplication of functions.

Kanjin [**3**] studied some properties of the algebra $A^{(\alpha,\tau)}$ and showed that

(B) [**3**, THEOREM 1, COROLLARY 1] The algebra $A^{(\alpha,\tau)}$ is semisimple and regular. The maximal ideal space of $A^{(\alpha,\tau)}$ is the interval $[0,\infty)$, and the Gelfand transform of f in $A^{(\alpha,\tau)}$ is given by f itself.

(C) [**3**, THEOREM 2] Let $x_0 > 0$. If $\alpha \geq 1/2$ and $\tau \geq 1$, then the singleton $\{x_0\}$ is not a set of spectral synthesis for $A^{(\alpha,\tau)}$.

Here, a closed set E of $[0,\infty)$ is called a *set of spectral synthesis* for $A^{(\alpha,\tau)}$ if a closed ideal I such that $Z(I) = E$ is unique, where $Z(I) = \{x \text{ in } [0,\infty); f(x) = 0$ for all f in $I\}$.

The purpose of this paper is to solve the problem which remains unsolved in (C).

THEOREM. (1) *Let $\alpha \geq -1/2$ and $\tau \geq 2$ or let $\alpha \geq \alpha_0$ and $\tau \geq 1$. Then, for every (α,τ), the singleton $\{0\}$ is a set of spectral synthesis for $A^{(\alpha,\tau)}$.*

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(2) *Let* $x_0 > 0$. *If* $-1/2 \le \alpha < 1/2$ *and* $\tau \ge 2$ *or if* $\alpha_0 \le \alpha < 1/2$ *and* $\tau \ge 1$, *then the singleton* $\{x_0\}$ *is a set of spectral synthesis for* $A^{(\alpha,\tau)}$.

This theorem is an immediate consequence of the following proposition which will be proved in §3.

PROPOSITION. *Let* $\alpha \ge -1/2$ *and* $\tau \ge 2$ *or let* $\alpha \ge \alpha_0$ *and* $\tau \ge 1$. *Let* I *be a closed ideal in* $A^{(\alpha,\tau)}$ *such that* $Z(I) = \{x_0\}$, $x_0 \ge 0$. *If* $x_0 > 0$, *then* $I = \{f \text{ in } A^{(\alpha,\tau)}; f^{(j)}(x_0) = 0,\ j = 0, 1, \dots, M\}$ *for some* $M \le \alpha + 1/2$. *If* $x_0 = 0$, *then* $I = \{f \text{ in } A^{(\alpha,\tau)}; f(0) = 0\}$.

Related results will be found in Cazzaniga and Meaney [**2**], Wolfenstetter [**8**], and Schwartz [**6**]. They are concerned with spectral synthesis on the algebra of absolutely convergent Jacobi polynomial series and on the algebra of Hankel transforms.

2. A lemma. First, we will prepare a lemma for the proof of the proposition. Let $C_c^\infty[0, \infty)$ be the space of functions on $[0, \infty)$ which are the restrictions of infinitely differentiable functions with compact support in $(-\infty, \infty)$.

LEMMA. *Let* $\alpha \ge -1/2$ *and* $\tau \ge 2$ *or let* $\alpha \ge \alpha_0$ *and* $\tau \ge 1$.

(1) *Let* f *be in* $C_c^\infty[0, \infty)$ *and let* q *be the least integer greater than* $\alpha + 3/2$. *Then* f *is in* $A^{(\alpha,\tau)}$ *and*

$$\|f\| \le C \left(\sup_{x \ge 0} |f(x)e^{\tau x}| + K^q \sup_{x \ge 0} \left| \left(\frac{d}{dx} \right)^q f(x)e^{\tau x} \right| \right),$$

where C *is a constant depending only on* α *and* τ, *and* K *is a number such that* $\operatorname{supp} f \subset [0, K]$.

(2) $C_c^\infty[0, \infty)$ *is dense in* $A^{(\alpha,\tau)}$.

(3) *Let* f *be in* $A^{(\alpha,\tau)}$ *and let* r *be the greatest integer not exceeding* $\alpha + 1/2$. *Then* f *is* r*-times continuously differentiable and, for* x *in* $(0, \infty)$ *and* $j = 0, 1, \dots, r$, *there exists a constant* B *not depending on* f *such that* $|f^{(j)}(x)| \le B\|f\|$.

PROOF. (2) is [**3**, Lemma 2] and (3) is implicitly proved in the proof of [**3**, Theorem 2], and also, in weak form, (1) is given in [**3**]. Here, we will only give an outline of the proof of (1). If $f(x) = \sum_{n=0}^\infty a_n R_n^\alpha(x) e^{-\tau x}$, then

$$a_n = \Gamma(\alpha + 1)^{-1} \int_0^\infty f(x)e^{\tau x} L_n^\alpha(x) e^{-x} x^\alpha \, dx.$$

We put $\|f\| = \left\{ \sum_{n \le 1/K} + \sum_{1/K < n} \right\} |a_n| = S_1 + S_2$. For S_1, we have

$$S_1 \le \frac{1}{\Gamma(\alpha + 1)} \sum_{n \le 1/K} \int_0^K |f(x)e^{\tau x}| \, |L_n^\alpha(x)| e^{-x/2} x^\alpha e^{-x/2} \, dx$$

and, by the inequality $|L_n^\alpha(x)| e^{-x/2} x^\alpha \le C$ for $0 < x \le 1/n$ (cf. [**7**, 8.22]), we have $S_1 \le C \sup_{0 \le x} |f(x)e^{\tau x}|$. Here and below, the letter C means positive constants depending only on α and τ, and it may vary from inequality to inequality. From integration by parts, it follows that

$$a_n = \frac{(n - q)!(-1)^q}{\Gamma(\alpha + 1) n!} \int_0^\infty \left\{ \left(\frac{d}{dx} \right)^q f(x)e^{\tau x} \right\} L_{n-q}^{\alpha+q}(x) e^{-x} x^{\alpha+q} \, dx.$$

We have

$$S_2 \le \sup_{0\le x}\left|\left(\frac{d}{dx}\right)^q f(x)e^{\tau x}\right| \sum_{1/K<n} n^{-q}\int_0^K |L_{n-q}^{\alpha+q}(x)|e^{-x}x^{\alpha+q}\,dx$$

$$\le C\sup_{0\le x}\left|\left(\frac{d}{dx}\right)^q f(x)e^{\tau x}\right|\left\{\sum_{1/K<n} n^{-q}\int_0^{1/n} + \sum_{1/K<n} n^{-q}\int_{1/n}^K\right\}$$

$$\le C\sup_{0\le x}\left|\left(\frac{d}{dx}\right)^q f(x)e^{\tau x}\right|\{I_1+I_2\},\quad \text{say.}$$

Then we have $I_1 \le CK^q$ and, by the inequality

$$|L_n^\alpha(x)| \le Ce^{-x/2}x^{-\alpha/2-1/4}n^{\alpha/2-1/4}$$

(cf. [**7**, 8.22]), we have $I_2 \le CK^q$. Q.E.D.

3. Proof of the proposition. Let L_I be the space of continuous linear functionals ϕ on $A^{(\alpha,\tau)}$ such that $\phi(f) = 0$ for all f in I. We will show that, if ϕ is in L_I, then ϕ is of the form

$$(*)\qquad \phi(f) = \begin{cases} \sum_{j=0}^p a_j\delta_{x_0}^{(j)}(f), & p\le \alpha+1/2\ (x_0>0),\\ a_0\delta_{x_0}(f), & (x_0=0)\end{cases}$$

for f in $A^{(\alpha,\tau)}$, where $\delta_{x_0}^{(j)}$ is the functional such that $\delta_{x_0}^{(j)}(f) = f^{(j)}(x_0)$ for f in $A^{(\alpha,\tau)}$. Then the proposition is proved as follows. Let $p(\phi) = \max\{j; a_j\ne 0\}$ for ϕ in L_I, and $M = \max\{p(\phi);\phi \text{ in } L_I\}$. By $(*)$, we have that $M=0$ for $x_0=0$ and $0\le M\le \alpha+1/2$ for $x_0>0$. Let ϕ_0 be a functional in L_I such that $M = p(\phi_0)$. From (1) in the lemma it follows that there exist functions h_m in $A^{(\alpha,\tau)}$ such that $h_m^{(k)}(x_0) = \delta_{mk}$, $k,m = 0,1,\ldots,M$, where δ_{mk} is Kronecker's symbol. For every f in I, we have

$$0 = \phi_0(fh_m) = \sum_{k=0}^M\left\{\sum_{j=k}^M {}_jC_k a_j f^{(j-k)}(x_0)\right\}h_m^{(k)}(x_0)$$

$$= \sum_{j=m}^M {}_jC_m a_j f^{(j-m)}(x_0),\qquad m = 0,1,\ldots,M.$$

Thus $f^{(j)}(x_0) = 0$ for $j = 0,1,\ldots,M$. This implies that $I = \{f \text{ in } A^{(\alpha,\tau)}; f^{(j)}(x_0) = 0, j = 0,1,\ldots,M\}$ since I is the space of f in $A^{(\alpha,\tau)}$ such that $\phi(f) = 0$ for all ϕ in L_I.

Now we will prove $(*)$. Let $D(-\infty,\infty)$ be the test function space on $(-\infty,\infty)$ with usual topology. For f in $D(-\infty,\infty)$, we put $f_P(x) = f(x)$, $x\ge 0$, and $f_N(x) = f(-x)$, $x\ge 0$. Then, by (1) in the lemma, we have tht f_P and f_N are in $A^{(\alpha,\tau)}$. Let ϕ be in L_I. We define $\Phi_+(f) = \phi(f_P)+\phi(f_N)$ and $\Phi_-(f) = \phi(f_P)-\phi(f_N)$ for f in $D(-\infty,\infty)$. By (1) again, we have

$$|\Phi_\pm(f)| \le \|\phi\|(\|f_P\|+\|f_N\|)$$

$$\le C\|\phi\|e^{\tau K}\left(\sup_{-\infty<x<\infty}|f(x)| + K^q\sum_{j=1}^q \sup_{-\infty<x<\infty}|f^{(j)}(x)|\right),$$

where K is a number such that $\operatorname{supp} f \subset [-K, K]$, and q is the least integer greater than $\alpha + 3/2$. Thus $\Phi_\pm$ are continuous linear functionals on $D(-\infty, \infty)$ with order not exceeding q. Since $A^{(\alpha,\tau)}$ is regular, the ideal I contains the ideal of functions in $A^{(\alpha,\tau)}$ which vanish on a neighborhood of x_0 (cf. [**4**, 5.7]). This implies that the supports of $\Phi_\pm$ are the singleton $\{x_0\}$. Thus $\Phi_\pm$ have the forms

$$\Phi_+ = \sum_{j=0}^{q} a_j^+ \delta_{x_0}^{(j)}, \qquad \Phi_- = \sum_{j=0}^{q} a_j^- \delta_{x_0}^{(j)},$$

where the $a_j^\pm$ are constants (cf. [**5**, 6.25]).

We will show that $a_j^\pm = 0$ for $j > \alpha + 1/2$ if $x_0 > 0$. Let $u(x)$ be a function in $D(-\infty, \infty)$ such that $u(x) = 1$ on a neighborhood of x_0 and $\operatorname{supp} u \subset (0, \infty)$. Then the function $u(x)e^{-\tau x}R_n^\alpha(x)$ is in $D(-\infty, \infty)$, and

$$|\Phi_\pm(ue^{-\tau x}R_n^\alpha)| \le |\phi(ue^{-\tau x}R_n^\alpha)| \le \|\phi\|\,\|u\|$$

since $\|(e^{-\tau x}R_n^\alpha)_P\| = 1$. In particular, $\Phi_\pm(ue^{-\tau x}R_n^\alpha) = O(1)$ $(n \to \infty)$. On the other hand, by the formula $(d/dx)L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x)$ (cf. [**7**, (5.1.14)]) and the asymptotic formula

$$L_n^\alpha(x) = \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4}n^{\alpha/2-1/4}\cos[2(nx)^{1/2} - \alpha\pi/2 - \pi/4] + O(n^{\alpha/2-3/4}), \qquad x > 0$$

(cf. [**7**, (8.22.1)]), we have

$$\delta_{x_0}^{(j)}(ue^{-\tau x}R_n^\alpha) = O(n^{-(\alpha-j)/2-1/4}) \qquad (n \to \infty),$$

and

$$\limsup_{n\to\infty} |\delta_{x_0}^{(j)}(ue^{-\tau x}R_n^\alpha)|n^{(\alpha-j)/2+1/4} > 0$$

for $j = 0, 1, \ldots, q$. This implies that

$$\limsup_{n\to\infty} |\Phi_\pm(ue^{-\tau x}R_n^\alpha)| = \infty$$

if $a_j^\pm \ne 0$ for some $j > \alpha + 1/2$. Thus we have $a_j^\pm = 0$ for $j > \alpha + 1/2$.

Next we will show that $a_j^\pm = 0$ for $j > 0$ if $x_0 = 0$. Let $u_1(x)$ be an even function in $D(-\infty, \infty)$ such that $u_1(x) = 1$ for x in $[-1/2, 1/2]$ and $u_1(x) = 0$ for x not in $(-1, 1)$. Put $u_n(x) = u_1(nx)$, $n = 2, 3, \ldots$, and consider the function $u_n(x)e^{-\tau x}R_n^\alpha(x)$, $-\infty < x < \infty$. Then we have

$$|\Phi_\pm(u_ne^{-\tau x}R_n^\alpha) \le |\phi((u_ne^{-\tau x}R_n^\alpha)_N)| + |\phi((u_ne^{-\tau x}R_n^\alpha)_P)|.$$

Since $\|(u_n)_P\| = O(1)$ $(n \to \infty)$ by (1) in the lemma and $\|(e^{-\tau x}R_n^\alpha)_P\| = 1$, we have

$$\begin{aligned}|\phi((u_ne^{-\tau x}R_n^\alpha)_P)| &\le \|\phi\|\,\|(u_ne^{-\tau x}R_n^\alpha)_P\| \\ &\le \|\phi\|\,\|(u_n)_P\|\,\|(e^{-\tau x}R_n^\alpha)_P\| = O(1) \qquad (n \to \infty).\end{aligned}$$

Moreover, we will claim that $\phi((u_ne^{-\tau x}R_n^\alpha)_N) = O(1)$ $(n \to \infty)$. By (1) again, we have

$$\|(u_ne^{-\tau x}R_n^\alpha)_N\| \le C\left(\sup_{-1/n\le x\le 0}|u_n(x)e^{-2\tau x}R_n^\alpha(x)| + n^{-q}\sup_{-1/n\le x\le 0}\left|\left(\frac{d}{dx}\right)^q u_n(x)e^{-2\tau x}R_n^\alpha(x)\right|\right).$$

We have that

$$\left(\frac{d}{dx}\right)^j u_n(x)R_n^\alpha(x) = \sum_{k=0}^{j} {}_jC_k n^{j-k} u_1^{(j-k)}(nx) \times \frac{(-1)^k\Gamma(n+1)\Gamma(\alpha+k+1)}{(\alpha+1)^k\Gamma(n+\alpha+1)} L_{n-k}^{\alpha+k}(x), \qquad j = 0, 1, 2, \ldots.$$

By Perron's formula in the complex domain (see [**7**, (8.22.3)]), we have

$$\sup_{-1/n\le x\le 0}\left|\left(\frac{d}{dx}\right)^j u_n(x)R_n^\alpha(x)\right| = O(n^j) \qquad (n\to\infty),\ j=0,1,2,\ldots,$$

and thus we have $\|(u_n e^{-\tau x}R_n^\alpha)_N\| = O(1)$ $(n\to\infty)$. Since

$$|\phi((u_n e^{-\tau x}R_n^\alpha)_N)| \le \|\phi\|\,\|(u_n e^{-\tau x}R_n^\alpha)_N\|,$$

we have the claim $\phi((u_n e^{-\tau x}R_n^\alpha)_N) = O(1)$ $(n\to\infty)$. Therefore, we have

$$\Phi_\pm(u_n e^{-\tau x}R_n^\alpha) = O(1) \qquad (n\to\infty).$$

On the other hand, we have

$$\begin{aligned}\Phi_\pm(u_n e^{-\tau x}R_n^\alpha) &= \sum_{j=0}^{q} a_j^\pm \delta_0^{(j)}(u_n e^{-\tau x}R_n^\alpha)\\ &= \sum_{j=0}^{q} a_j^\pm(-\tau)^{-j}\sum_{k=0}^{j} {}_jC_k\tau^k\frac{n(n-1)\cdots(n-k+1)}{(\alpha+1)^k}.\end{aligned}$$

This implies that, if $a_j^\pm \ne 0$ for some $j>0$, then

$$\lim_{n\to\infty}|\Phi_\pm(u_n e^{-\tau x}R_n^\alpha)| = \infty.$$

Thus we have that $a_j^\pm = 0$ for $j>0$, and therefore we have that $\phi(f_P)$ is of the form

$$\begin{aligned}\phi(f_P) &= (\Phi_+(f)+\Phi_-(f))/2\\ &= \begin{cases}\sum_{j=0}^{p} a_j\delta_{x_0}^{(j)}(f), & p\le\alpha+1/2\ (x_0>0),\\ a_0\delta_{x_0}(f), & (x_0=0)\end{cases}\end{aligned}$$

for f in $D(-\infty,\infty)$. From (2) and (3) in the lemma and $|f(0)|\le\|f\|$, it follows that ϕ is of the form $(*)$. Q.E.D.

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