SPECTRAL SYNTHESIS ON THE ALGEBRA OF ABSOLUTELY CONVERGENT LAGUERRE POLYNOMIAL SERIES

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ABSTRACT. Askey and Gasper [1] constructed the algebra with convolution structure for Laguerre polynomials. We will answer the question of spectral synthesis of the one point on this algebra.

1. Introduction. Let $L_n^\alpha(x)$ be the Laguerre polynomial given by

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \left( \frac{d}{dx} \right)^n [e^{-x} x^{n+\alpha}],$$

and denote by $R_n^\alpha(x)$ the normalized Laguerre polynomial so that

$$R_n^\alpha(x) = L_n^\alpha(x)/L_n^\alpha(0),$$

where $\alpha > -1$ and $n$ is a nonnegative integer.

Let $\alpha \geq -1/2$ and $\tau \geq 2$ or let $\alpha > \alpha_0 = (-5 + (17)^{1/2})/2$ and $\tau \geq 1$. Let $A^{(\alpha, \tau)}$ be the space

$$\left\{ f(x) \text{ on } [0, \infty); \right. \left. f(x) = \sum_{n=0}^\infty a_n R_n^\alpha(x) e^{-\tau x}, \sum_{n=0}^\infty |a_n| < \infty \right\},$$

and introduce a norm to $A^{(\alpha, \tau)}$ by $\|f\| = \sum_{n=0}^\infty |a_n|$. Then Askey and Gasper [1] showed that

(A) [1, §§4, 5] $A^{(\alpha, \tau)}$ is a Banach algebra of continuous functions on the interval $[0, \infty)$ vanishing at infinity with the product of pointwise multiplication of functions.

Kanjin [3] studied some properties of the algebra $A^{(\alpha, \tau)}$ and showed that

(B) [3, THEOREM 1, COROLLARY 1] The algebra $A^{(\alpha, \tau)}$ is semisimple and regular. The maximal ideal space of $A^{(\alpha, \tau)}$ is the interval $[0, \infty)$, and the Gelfand transform of $f$ in $A^{(\alpha, \tau)}$ is given by $f$ itself.

(C) [3, THEOREM 2] Let $x_0 > 0$. If $\alpha \geq 1/2$ and $\tau \geq 1$, then the singleton $\{x_0\}$ is not a set of spectral synthesis for $A^{(\alpha, \tau)}$.

Here, a closed set $E$ of $[0, \infty)$ is called a set of spectral synthesis for $A^{(\alpha, \tau)}$ if a closed ideal $I$ such that $Z(I) = E$ is unique, where $Z(I) = \{x \text{ in } [0, \infty); f(x) = 0 \text{ for all } f \text{ in } I\}$.

The purpose of this paper is to solve the problem which remains unsolved in (C).

THEOREM. (1) Let $\alpha \geq -1/2$ and $\tau \geq 2$ or let $\alpha > \alpha_0$ and $\tau \geq 1$. Then, for every $(\alpha, \tau)$, the singleton $\{0\}$ is a set of spectral synthesis for $A^{(\alpha, \tau)}$.
(2) Let \( x_0 > 0 \). If \(-1/2 \leq \alpha < 1/2\) and \( \tau \geq 2\) or if \( \alpha_0 \leq \alpha < 1/2\) and \( \tau \geq 1\), then the singleton \( \{ x_0 \} \) is a set of spectral synthesis for \( A^{(\alpha, \tau)} \).

This theorem is an immediate consequence of the following proposition which will be proved in §3.

**Proposition.** Let \( \alpha \geq -1/2\) and \( \tau \geq 2\) or let \( \alpha \geq \alpha_0\) and \( \tau \geq 1\). Let \( I \) be a closed ideal in \( A^{(\alpha, \tau)} \) such that \( \mathbb{Z}(I) = \{ x_0 \}, x_0 \geq 0 \). If \( x_0 > 0 \), then \( I = \{ f \in A^{(\alpha, \tau)}; f^{(j)}(x_0) = 0, j = 0, 1, \ldots, M \} \) for some \( M \leq \alpha + 1/2\). If \( x_0 = 0 \), then \( I = \{ f \in A^{(\alpha, \tau)}; f(0) = 0 \} \).

Related results will be found in Cazzaniga and Meaney [2], Wolfenstetter [8], and Schwartz [6]. They are concerned with spectral synthesis on the algebra of absolutely convergent Jacobi polynomial series and on the algebra of Hankel transforms.

2. **A lemma.** First, we will prepare a lemma for the proof of the proposition. Let \( C_c^\infty[0, \infty) \) be the space of functions on \([0, \infty)\) which are the restrictions of infinitely differentiable functions with compact support in \((-\infty, \infty)\).

**Lemma.** Let \( \alpha \geq -1/2\) and \( \tau \geq 2\) or let \( \alpha \geq \alpha_0\) and \( \tau \geq 1\).

1. Let \( f \) be in \( C_c^\infty[0, \infty) \) and let \( q \) be the least integer greater than \( \alpha + 3/2\). Then \( f \) is in \( A^{(\alpha, \tau)} \) and

\[
\|f\| \leq C \left( \sup_{x \geq 0} |f(x)e^{\tau x}| + K^q \sup_{x \geq 0} \left| \left( \frac{d}{dx} \right)^q f(x)e^{\tau x} \right| \right),
\]

where \( C \) is a constant depending only on \( \alpha \) and \( \tau \), and \( K \) is a number such that \( \sup f \subset [0, K] \).

2. \( C_c^\infty[0, \infty) \) is dense in \( A^{(\alpha, \tau)} \).

3. Let \( f \) be in \( A^{(\alpha, \tau)} \) and let \( r \) be the greatest integer not exceeding \( \alpha + 1/2 \). Then \( f \) is \( r \)-times continuously differentiable and, for \( x \) in \((0, \infty)\) and \( j = 0, 1, \ldots, r \), there exists a constant \( B \) not depending on \( f \) such that \( |f^{(j)}(x)| \leq B\|f\| \).

**Proof.** (2) is [3, Lemma 2] and (3) is implicitly proved in the proof of [3, Theorem 2], and also, in weak form, (1) is given in [3]. Here, we will only give an outline of the proof of (1). If \( f(x) = \sum_{n=0}^{\infty} a_n R_n^\alpha(x)e^{-\tau x} \), then

\[
a_n = \Gamma(\alpha + 1)^{-1} \int_0^\infty f(x)e^{\tau x}L_n^\alpha(x)e^{-x^\alpha} \, dx.
\]

We put \( \|f\| = \left\{ \sum_{n \leq 1/K} \left| a_n \right| + \sum_{1/K < n < K} \right\} |a_n| = S_1 + S_2 \). For \( S_1 \), we have

\[
S_1 \leq \frac{1}{\Gamma(\alpha + 1)} \sum_{n \leq 1/K} \int_0^K \left| f(x)e^{\tau x} \right| |L_n^\alpha(x)|e^{-x^2}e^{-x^2/\tau} \, dx
\]

and, by the inequality \( |L_n^\alpha(x)|e^{-x^2/\tau} \leq C \) for \( 0 < x < 1/n \) (cf. [7, 8.22]), we have \( S_1 \leq C \sup_{0 \leq x \leq 1/n} |f(x)e^{\tau x}| \). Here and below, the letter \( C \) means positive constants depending only on \( \alpha \) and \( \tau \), and it may vary from inequality to inequality. From integration by parts, it follows that

\[
a_n = \frac{(n-q)!(\alpha+q)!}{\Gamma(n+q+1)!} \int_0^\infty \left\{ \left( \frac{d}{dx} \right)^q f(x)e^{\tau x} \right\} L_{n-q}^{\alpha+q}(x)e^{-x^\alpha} \, dx.
\]
We have
\[
S_2 \leq \sup_{0 \leq x} \left| \left( \frac{d}{dx} \right)^q f(x) e^{rx} \right| \sum_{1/K < n} n^{-q} \int_0^K |L_{n-q}^a(x)| e^{-axq} \, dx
\]
\[
\leq C \sup_{0 \leq x} \left| \left( \frac{d}{dx} \right)^q f(x) e^{rx} \right| \left\{ \sum_{1/K < n} n^{-q} \int_0^{1/n} + \sum_{1/K < n} n^{-q} \int_{1/n}^K \right\}
\]
\[
\leq C \sup_{0 \leq x} \left| \left( \frac{d}{dx} \right)^q f(x) e^{rx} \right| \{I_1 + I_2\}, \text{ say.}
\]
Then we have \( I_1 \leq CK^q \) and, by the inequality
\[
|L_n^a(x)| \leq Ce^{-x/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4}
\]
(cf. [7, 8.22]), we have \( I_2 \leq CK^q \). Q.E.D.

3. Proof of the proposition. Let \( L_I \) be the space of continuous linear functionals \( \phi \) on \( A^{(\alpha, \tau)} \) such that \( \phi(f) = 0 \) for all \( f \) in \( I \). We will show that, if \( \phi \) is in \( L_I \), then \( \phi \) is of the form

\[
\phi(f) = \begin{cases} 
\sum_{j=0}^p a_j \delta_{x_0}^{(j)}(f), & p \leq \alpha + 1/2 \ (x_0 > 0), \\
0 & (x_0 = 0)
\end{cases}
\]

for \( f \) in \( A^{(\alpha, \tau)} \), where \( \delta_{x_0}^{(j)} \) is the functional such that \( \delta_{x_0}^{(j)}(f) = f^{(j)}(x_0) \) for \( f \) in \( A^{(\alpha, \tau)} \). Then the proposition is proved as follows. Let \( p(\phi) = \max\{j; a_j \neq 0\} \) for \( \phi \) in \( L_I \), and \( M = \max\{p(\phi); \phi \in L_I\} \). By (\ast\ast\ast\ast\), we have that \( M = 0 \) for \( x_0 = 0 \) and \( 0 \leq M \leq \alpha + 1/2 \) for \( x_0 > 0 \). Let \( \phi_0 \) be a functional in \( L_I \) such that \( M = p(\phi_0) \).

From (1) in the lemma it follows that there exist functions \( h_m \) in \( A^{(\alpha, \tau)} \) such that \( h_m^{(k)}(x_0) = \delta_{mk} \), \( k, m = 0, 1, \ldots, M \), where \( \delta_{mk} \) is Kronecker’s symbol. For every \( f \) in \( I \), we have

\[
0 = \phi_0(fh_m) = \sum_{k=0}^M \left\{ \sum_{j=k}^M jC_m a_j f^{(j-k)}(x_0) \right\} h_m^{(k)}(x_0)
\]

\[
= \sum_{j=m}^M jC_m a_j f^{(j-m)}(x_0), \quad m = 0, 1, \ldots, M.
\]

Thus \( f^{(j)}(x_0) = 0 \) for \( j = 0, 1, \ldots, M \). This implies that \( I = \{ f \in A^{(\alpha, \tau)} ; f^{(j)}(x_0) = 0, j = 0, 1, \ldots, M \} \) since \( I \) is the space of \( f \) in \( A^{(\alpha, \tau)} \) such that \( \phi(f) = 0 \) for all \( \phi \) in \( L_I \).

Now we will prove (\ast\ast\ast\ast\ast). Let \( D(-\infty, \infty) \) be the test function space on \((-\infty, \infty)\) with usual topology. For \( f \) in \( D(-\infty, \infty) \), we put \( f_P(x) = f(x), \ x \geq 0 \), and \( f_N(x) = f(-x), \ x \geq 0 \). Then, by (1) in the lemma, we have that \( f_P \) and \( f_N \) are in \( A^{(\alpha, \tau)} \). Let \( \phi \) be in \( L_I \). We define \( \Phi_+ = \phi(f_P) + \phi(f_N) \) and \( \Phi_- = \phi(f_P) - \phi(f_N) \) for \( f \) in \( D(-\infty, \infty) \). By (1) again, we have

\[
|\Phi_\pm(f)| \leq \|\phi\| \left( \|f_P\| + \|f_N\| \right)
\]

\[
\leq C\|\phi\| e^{rK} \left( \sup_{-\infty < x < \infty} |f(x)| + K^q \sum_{j=1}^q \sup_{-\infty < x < \infty} |f^{(j)}(x)| \right),
\]
where \( K \) is a number such that \( \text{supp} \ f \subset [-K, K] \), and \( q \) is the least integer greater
than \( \alpha + 3/2 \). Thus \( \Phi_{\pm} \) are continuous linear functionals on \( D(-\infty, \infty) \) with order
not exceeding \( q \). Since \( A^{(\alpha, \gamma)} \) is regular, the ideal \( I \) contains the ideal of functions
in \( A^{(\alpha, \gamma)} \) which vanish on a neighborhood of \( x_0 \) (cf. \([4, 5.7]\)). This implies that the
supports of \( \Phi_{\pm} \) are the singleton \( \{x_0\} \). Thus \( \Phi_{\pm} \) have the forms
\[
\Phi_+ = \sum_{j=0}^{q} a_j^+ \delta_{x_0}^{(j)}, \quad \Phi_- = \sum_{j=0}^{q} a_j^- \delta_{x_0}^{(j)},
\]
where the \( a_j^\pm \) are constants (cf. \([5, 6.25]\)).

We will show that \( a_j^\pm = 0 \) for \( j > \alpha + 1/2 \) if \( x_0 > 0 \). Let \( u(x) \) be a function in
\( D(-\infty, \infty) \) such that \( u(x) = 1 \) on a neighborhood of \( x_0 \) and \( \text{supp} \ u \subset (0, \infty) \). Then
the function \( u(x)e^{-\tau x R_n^\alpha(x)} \) is in \( D(-\infty, \infty) \), and
\[
|\Phi_{\pm}(ue^{-\tau x R_n^\alpha})| \leq |\phi(ue^{-\tau x R_n^\alpha})| \leq ||\phi|| \|u\|
\]
since \( \|(e^{-\tau x R_n^\alpha})_p\| = 1 \). In particular, \( \Phi_{\pm}(ue^{-\tau x R_n^\alpha}) = O(1) \) \( (n \to \infty) \). On the
other hand, by the formula \( (d/dx)L_n^{\alpha}(x) = -L_n^{\alpha+1}(x) \) (cf. \([7, 5.1.14]\)) and the
asymptotic formula
\[
L_n^{\alpha}(x) = \pi^{-1/2} e^{-2x/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4} \cos[2(nx)^{1/2} - \alpha \pi/2 - \pi/4]
+ O(n^{\alpha/2-3/4}), \quad x > 0
\]
(cf. \([7, 8.22.1]\)), we have
\[
|\delta_{x_0}^{(j)}(ue^{-\tau x R_n^\alpha})| = O(n^{-(\alpha-j)/2-1/4}) \quad (n \to \infty),
\]
and
\[
\limsup_{n \to \infty} |\delta_{x_0}^{(j)}(ue^{-\tau x R_n^\alpha})| n^{(\alpha-j)/2+1/4} > 0
\]
for \( j = 0, 1, \ldots, q \). This implies that
\[
\limsup_{n \to \infty} |\Phi_{\pm}(ue^{-\tau x R_n^\alpha})| = \infty
\]
if \( a_j^\pm \neq 0 \) for some \( j > \alpha + 1/2 \). Thus we have \( a_j^\pm = 0 \) for \( j > \alpha + 1/2 \).

Next we will show that \( a_j^\pm = 0 \) for \( j > 0 \) if \( x_0 = 0 \). Let \( u_1(x) \) be an even
function in \( D(-\infty, \infty) \) such that \( u_1(x) = 1 \) for \( x \) in \([-1/2, 1/2]\) and \( u_1(x) = 0 \) for
\( x \) not in \((-1, 1)\). Put \( u_n(x) = u_1(nx), \ n = 2, 3, \ldots, \) and consider the function
\( u_n(x)e^{-\tau x R_n^\alpha(x)}, \ -\infty < x < \infty \). Then we have
\[
|\Phi_{\pm}(u_n e^{-\tau x R_n^\alpha})| \leq |\phi((u_n e^{-\tau x R_n^\alpha})_N)| + |\phi((u_n e^{-\tau x R_n^\alpha})_p)|.
\]
Since \( \|(u_n)_p\| = O(1) \) \( (n \to \infty) \) by \( (1) \) in the lemma and \( \|(e^{-\tau x R_n^\alpha})_p\| = 1 \), we have
\[
|\phi((u_n e^{-\tau x R_n^\alpha})_p)| \leq \|\phi\| \|(u_n e^{-\tau x R_n^\alpha})_p\|
\]
\[
\leq \|\phi\| \|(u_n)_p\| \|(e^{-\tau x R_n^\alpha})_p\| = O(1) \quad (n \to \infty).
\]
Moreover, we will claim that \( \phi((u_n e^{-\tau x R_n^\alpha})_N) = O(1) \) \( (n \to \infty) \). By \( (1) \) again, we have
\[
\|(u_n e^{-\tau x R_n^\alpha})_N\| \leq C \left( \sup_{-1/n \leq x \leq 0} |u_n(x)e^{-\tau x R_n^\alpha(x)}| \right.
\]
\[
\left. + n^{-q} \sup_{-1/n \leq x \leq 0} \left| \left( \frac{d}{dx} \right)^q u_n(x)e^{-\tau x R_n^\alpha(x)} \right| \right).
\]
We have that
\[
\left( \frac{d}{dx} \right)^j u_n(x) R_n^\alpha(x) = \sum_{k=0}^{j} j C_k n^{j-k} u_1^{(j-k)}(nx) \\
\times \frac{(-1)^k \Gamma(n+1) \Gamma(\alpha+k+1)}{\Gamma(\alpha+1) \Gamma(n+\alpha+1)} L_{n-k}^{\alpha+k}(x), \quad j = 0, 1, 2, \ldots.
\]

By Perron’s formula in the complex domain (see [7, (8.22.3)]), we have
\[
\sup_{-1/n \leq x \leq 0} \left| \left( \frac{d}{dx} \right)^j u_n(x) R_n^\alpha(x) \right| = O(n^j) \quad (n \to \infty), \quad j = 0, 1, 2, \ldots,
\]
and thus we have \(\|u_n e^{-\tau x} R_n^\alpha\| = O(1) \quad (n \to \infty)\). Since
\[
|\phi(u_n e^{-\tau x} R_n^\alpha)| \leq \|\phi\| \|u_n e^{-\tau x} R_n^\alpha\|,
\]
we have the claim \(\phi(u_n e^{-\tau x} R_n^\alpha) = O(1) \quad (n \to \infty)\). Therefore, we have
\[
\Phi_\pm(u_n e^{-\tau x} R_n^\alpha) = O(1) \quad (n \to \infty).
\]

On the other hand, we have
\[
\Phi_\pm(u_n e^{-\tau x} R_n^\alpha) = \sum_{j=0}^{q} a_j^\pm \delta_0^{(j)}(u_n e^{-\tau x} R_n^\alpha)
\]
\[
= \sum_{j=0}^{q} a_j^\pm (-\tau)^{-j} \sum_{k=0}^{j} j C_k r^k \frac{n(n-1) \cdots (n-k+1)}{(\alpha+1)^k}.
\]

This implies that, if \(a_j^\pm \neq 0\) for some \(j > 0\), then
\[
\lim_{n \to \infty} |\Phi_\pm(u_n e^{-\tau x} R_n^\alpha)| = \infty.
\]

Thus we have that \(a_j^\pm = 0\) for \(j > 0\), and therefore we have that \(\phi(f_P)\) is of the form
\[
\phi(f_P) = (\Phi_+(f) + \Phi_-(f))/2
\]
\[
= \left\{ \begin{array}{ll}
\sum_{j=0}^{p} a_j \delta_{x_0}^{(j)}(f), & p \leq \alpha + 1/2 \quad (x_0 > 0), \\
a_0 \delta_{x_0}(f), & (x_0 = 0)
\end{array} \right.
\]
for \(f\) in \(D(-\infty, \infty)\). From (2) and (3) in the lemma and \(|f(0)| \leq \|f\|\), it follows that \(\phi\) is of the form \((\star)\). Q.E.D.

REFERENCES


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