SPECTRAL SYNTHESIS ON THE ALGEBRA OF ABSOLUTELY
CONVERGENT LAGUERRE POLYNOMIAL SERIES
YÜICHI KANJIN

ABSTRACT. Askey and Gasper [1] constructed the algebra with convolution
structure for Laguerre polynomials. We will answer the question of spectral
synthesis of the one point on this algebra.

1. Introduction. Let \( L_n^\alpha(x) \) be the Laguerre polynomial given by
\[
L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \left( \frac{d}{dx} \right)^n \left[ e^{-x} x^{n+\alpha} \right],
\]
and denote by \( R_n^\alpha(x) \) the normalized Laguerre polynomial so that
\[
R_n^\alpha(x) = \frac{L_n^\alpha(x)}{L_n^\alpha(0)},
\]
where \( \alpha > -1 \) and \( n \) is a nonnegative integer.

Let \( \alpha > -1/2 \) and \( \tau \geq 2 \) or let \( \alpha > \alpha_0 = (-5 + (17)^{1/2})/2 \) and \( \tau \geq 1 \). Let \( A^{(\alpha,\tau)} \)
be the space
\[
\left\{ f(x) \text{ on } [0, \infty); f(x) = \sum_{n=0}^{\infty} a_n R_n^\alpha(x) e^{-\tau x}, \sum_{n=0}^{\infty} |a_n| < \infty \right\},
\]
and introduce a norm to \( A^{(\alpha,\tau)} \) by \( \|f\| = \sum_{n=0}^{\infty} |a_n| \). Then Askey and Gasper [1]
showed that

(A) [1, §§4, 5] \( A^{(\alpha,\tau)} \) is a Banach algebra of continuous functions on the interval
\([0, \infty)\) vanishing at infinity with the product of pointwise multiplication of functions.

Kanjin [3] studied some properties of the algebra \( A^{(\alpha,\tau)} \) and showed that

(B) [3, THEOREM 1, COROLLARY 1] The algebra \( A^{(\alpha,\tau)} \) is semisimple and
regular. The maximal ideal space of \( A^{(\alpha,\tau)} \) is the interval \([0, \infty)\), and the Gelfand
transform of \( f \) in \( A^{(\alpha,\tau)} \) is given by \( f \) itself.

(C) [3, THEOREM 2] Let \( x_0 > 0 \). If \( \alpha \geq 1/2 \) and \( \tau \geq 1 \), then the singleton \( \{x_0\} \)
is not a set of spectral synthesis for \( A^{(\alpha,\tau)} \).

Here, a closed set \( E \) of \([0, \infty)\) is called a set of spectral synthesis for \( A^{(\alpha,\tau)} \) if a
closed ideal \( I \) such that \( Z(I) = E \) is unique, where \( Z(I) = \{x \in [0, \infty); f(x) = 0 \}
for all \( f \) in \( I \} \).

The purpose of this paper is to solve the problem which remains unsolved in (C).

THEOREM. (1) Let \( \alpha \geq -1/2 \) and \( \tau \geq 2 \) or let \( \alpha \geq \alpha_0 \) and \( \tau \geq 1 \). Then, for
every \( (\alpha, \tau) \), the singleton \( \{0\} \) is a set of spectral synthesis for \( A^{(\alpha,\tau)} \).
(2) Let $x_0 > 0$. If $-1/2 \leq \alpha < 1/2$ and $\tau \geq 2$ or if $\alpha_0 \leq \alpha < 1/2$ and $\tau \geq 1$, then the singleton $\{x_0\}$ is a set of spectral synthesis for $A^{(\alpha, \tau)}$.

This theorem is an immediate consequence of the following proposition which will be proved in §3.

**Proposition.** Let $\alpha \geq -1/2$ and $\tau \geq 2$ or let $\alpha \geq \alpha_0$ and $\tau \geq 1$. Let $I$ be a closed ideal in $A^{(\alpha, \tau)}$ such that $Z(I) = \{x_0\}$, $x_0 \geq 0$. If $x_0 > 0$, then $I = \{f \in A^{(\alpha, \tau)}; f^{(j)}(x_0) = 0, j = 0, 1, \ldots, M\}$ for some $M \leq \alpha + 1/2$. If $x_0 = 0$, then $I = \{f \in A^{(\alpha, \tau)}; f(0) = 0\}$.

Related results will be found in Cazzaniga and Meaney [2], Wolfenstetter [8], and Schwartz [6]. They are concerned with spectral synthesis on the algebra of absolutely convergent Jacobi polynomial series and on the algebra of Hankel transforms.

2. A lemma. First, we will prepare a lemma for the proof of the proposition. Let $C^\infty_c[0, \infty)$ be the space of functions on $[0, \infty)$ which are the restrictions of infinitely differentiable functions with compact support in $(-\infty, \infty)$.

**Lemma.** Let $\alpha \geq -1/2$ and $\tau \geq 2$ or let $\alpha \geq \alpha_0$ and $\tau \geq 1$.

(1) Let $f$ be in $C^\infty_c[0, \infty)$ and let $q$ be the least integer greater than $\alpha + 3/2$. Then $f$ is in $A^{(\alpha, \tau)}$ and

$$
\|f\| \leq C \left( \sup_{x \geq 0} |f(x)e^{\tau x}| + K^q \sup_{x \geq 0} \left( \frac{d}{dx} \right)^q f(x)e^{\tau x} \right),
$$

where $C$ is a constant depending only on $\alpha$ and $\tau$, and $K$ is a number such that $\text{supp} f \subset [0, K]$.

(2) $C^\infty_c[0, \infty)$ is dense in $A^{(\alpha, \tau)}$.

(3) Let $f$ be in $A^{(\alpha, \tau)}$ and let $r$ be the greatest integer not exceeding $\alpha + 1/2$. Then $f$ is $r$-times continuously differentiable and, for $x$ in $(0, \infty)$ and $j = 0, 1, \ldots, r$, there exists a constant $B$ not depending on $f$ such that $|f^{(j)}(x)| \leq B\|f\|$.

**Proof.** (2) is [3, Lemma 2] and (3) is implicitly proved in the proof of [3, Theorem 2], and also, in weak form, (1) is given in [3]. Here, we will only give an outline of the proof of (1). If $f(x) = \sum_{n=0}^{\infty} a_n R_n^\alpha(x)e^{-x^\alpha}$, then

$$
a_n = \Gamma(\alpha + 1)^{-1} \int_0^\infty f(x)e^{\tau x} L_n^\alpha(x)e^{-x^\alpha} dx.
$$

We put $\|f\| = \left\{ \sum_{n \leq 1/K} + \sum_{1/K < n} \right\} |a_n| = S_1 + S_2$. For $S_1$, we have

$$
S_1 \leq \frac{1}{\Gamma(\alpha + 1)} \sum_{n \leq 1/K} \int_0^K |f(x)e^{\tau x}| |L_n^\alpha(x)|e^{-x^\alpha/2}e^{-x^\alpha/2} dx
$$

and, by the inequality $|L_n^\alpha(x)|e^{-x^\alpha/2} \leq C$ for $0 < x \leq 1/n$ (cf. [7, 8.22]), we have $S_1 \leq C \sup_{0 \leq x} |f(x)e^{\tau x}|$. Here and below, the letter $C$ means positive constants depending only on $\alpha$ and $\tau$, and it may vary from inequality to inequality. From integration by parts, it follows that

$$
a_n = \frac{(n-q)!(1-q)^q}{\Gamma(\alpha + 1)n!} \int_0^\infty \left\{ \left( \frac{d}{dx} \right)^q f(x)e^{\tau x} \right\} L_n^{\alpha+q}(x)e^{-x^\alpha+q} dx.
$$
We have
\[
S_2 \leq \sup_{0 \leq x} \left| \left( \frac{d}{dx} \right)^q f(x) e^{rx} \right| \sum_{1/K \leq n} n^{-q} \int_0^K |L_{n-q}^q(x)| e^{-rx} x^{\alpha+q} \, dx
\]
\[
\leq C \sup_{0 \leq x} \left| \left( \frac{d}{dx} \right)^q f(x) e^{rx} \right| \left\{ \sum_{1/K \leq n} n^{-q} \int_0^{1/n} + \sum_{1/K \leq n} n^{-q} \int_{1/n}^K \right\}
\]
\[
\leq C \sup_{0 \leq x} \left| \left( \frac{d}{dx} \right)^q f(x) e^{rx} \right| \{I_1 + I_2\}, \text{ say.}
\]
Then we have \( I_1 \leq CKq \) and, by the inequality
\[
|L_n^q(x)| \leq Ce^{-x^2/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4}
\]
(cf. [7, 8.22]), we have \( I_2 \leq CKq \). Q.E.D.

3. Proof of the proposition. Let \( L_I \) be the space of continuous linear functionals \( \phi \) on \( A^{(\alpha, \tau)} \) such that \( \phi(f) = 0 \) for all \( f \) in \( I \). We will show that, if \( \phi \) is in \( L_I \), then \( \phi \) is of the form
\[
(*) \quad \phi(f) = \begin{cases} 
\sum_{j=0}^p a_j \delta_{x_0}^{(j)}(f), & p \leq \alpha + 1/2 \ (x_0 > 0), \\
0 & (x_0 = 0)
\end{cases}
\]
for \( f \) in \( A^{(\alpha, \tau)} \), where \( \delta_{x_0}^{(j)} \) is the functional such that \( \delta_{x_0}^{(j)}(f) = f^{(j)}(x_0) \) for \( f \) in \( A^{(\alpha, \tau)} \). Then the proposition is proved as follows. Let \( p(\phi) = \max\{j; a_j \neq 0\} \) for \( \phi \) in \( L_I \), and \( M = \max\{p(\phi); \phi \in L_I\} \). By (\(*)(\), we have that \( M = 0 \) for \( x_0 = 0 \) and \( 0 \leq M \leq \alpha + 1/2 \) for \( x_0 > 0 \). Let \( \phi_0 \) be a functional in \( L_I \) such that \( M = p(\phi_0) \).

From (1) in the lemma it follows that there exist functions \( h_m \) in \( A^{(\alpha, \tau)} \) such that \( h_m^{(k)}(x_0) = \delta_{mk}, \ k, m = 0, 1, \ldots, M \), where \( \delta_{mk} \) is Kronecker's symbol. For every \( f \) in \( I \), we have
\[
0 = \phi_0(fh_m) = \sum_{k=0}^M \left\{ \sum_{j=k}^M j C_k a_j f^{(j-k)}(x_0) \right\} h_m^{(k)}(x_0)
\]
\[
= \sum_{j=m}^M j C_m a_j f^{(j-m)}(x_0), \quad m = 0, 1, \ldots, M.
\]
Thus \( f^{(j)}(x_0) = 0 \) for \( j = 0, 1, \ldots, M \). This implies that \( I = \{ f \in A^{(\alpha, \tau)}; f^{(j)}(x_0) = 0, j = 0, 1, \ldots, M \} \) since \( I \) is the space of \( f \) in \( A^{(\alpha, \tau)} \) such that \( \phi(f) = 0 \) for all \( \phi \) in \( L_I \).

Now we will prove (\(*\)). Let \( D(-\infty, \infty) \) be the test function space on \( (-\infty, \infty) \) with usual topology. For \( f \) in \( D(-\infty, \infty) \), we put \( f_\tau(x) = f(x), \ x \geq 0 \), and \( f_N(x) = f(-x), \ x \geq 0 \). Then, by (1) in the lemma, we have that \( f_\tau \) and \( f_N \) are in \( A^{(\alpha, \tau)} \). Let \( \phi \) be in \( L_I \). We define \( \Phi_+(f) = \phi(f_\tau) + \phi(f_N) \) and \( \Phi_-(f) = \phi(f_\tau) - \phi(f_N) \) for \( f \) in \( D(-\infty, \infty) \). By (1) again, we have
\[
|\Phi_\pm(f)| \leq \|\phi\| (\|f_\tau\| + \|f_N\|)
\]
\[
\leq C \|\phi\| e^{K} \left( \sup_{-\infty < x < \infty} |f(x)| + K^q \sum_{j=1}^{q} \sup_{-\infty < x < \infty} |f^{(j)}(x)| \right),
\]
where $K$ is a number such that $\text{supp} \ f \subset [-K, K]$, and $q$ is the least integer greater than $\alpha + 3/2$. Thus $\Phi_{\pm}$ are continuous linear functionals on $D(-\infty, \infty)$ with order not exceeding $q$. Since $A^{(\alpha, \tau)}$ is regular, the ideal $I$ contains the ideal of functions in $A^{(\alpha, \tau)}$ which vanish on a neighborhood of $x_0$ (cf. [4, 5.7]). This implies that the supports of $\Phi_{\pm}$ are the singleton $\{x_0\}$. Thus $\Phi_{\pm}$ have the forms

$$\Phi_+ = \sum_{j=0}^{q} a_j^+ \delta_{x_0}^{(j)}, \quad \Phi_- = \sum_{j=0}^{q} a_j^- \delta_{x_0}^{(j)},$$

where the $a_j^\pm$ are constants (cf. [5, 6.25]).

We will show that $a_j^\pm = 0$ for $j > \alpha + 1/2$ if $x_0 > 0$. Let $u(x)$ be a function in $D(-\infty, \infty)$ such that $u(x) = 1$ on a neighborhood of $x_0$ and $\text{supp} \ u \subset (0, \infty)$. Then the function $u(x)e^{-\tau x^\alpha}_n(x)$ is in $D(-\infty, \infty)$, and

$$|\Phi_{\pm}(ue^{-\tau x^\alpha}_n)| \leq |\phi(u e^{-\tau x^\alpha}_n)| \leq ||\phi|| \|u||$$

since $||(e^{-\tau x^\alpha}_n)p|| = 1$. In particular, $\Phi_{\pm}(ue^{-\tau x^\alpha}_n) = O(1)$ $(n \to \infty)$. On the other hand, by the formula $(d/dx)L_n^{(\alpha)}(x) = -L_n^{(\alpha-1)}(x)$ (cf. [7, (5.1.14)]) and the asymptotic formula

$$L_n^{(\alpha)}(x) = \pi^{-1/2} e^{\alpha/2 x^2} (1 - \alpha/2 - \alpha x/2 - x^2/4) + O(n^{\alpha/2-3/4}), \quad x > 0$$

(cf. [7, (8.22.1)]), we have

$$\delta_{x_0}^{(j)}(ue^{-\tau x^\alpha}_n) = O(n^{-(\alpha-j)/2-1/4}) \quad (n \to \infty),$$

and

$$\lim_{n \to \infty} \sup_{n} |\delta_{x_0}^{(j)}(ue^{-\tau x^\alpha}_n)n^{(\alpha-j)/2+1/4} > 0$$

for $j = 0, 1, \ldots, q$. This implies that

$$\lim_{n \to \infty} \sup_{n} |\Phi_{\pm}(ue^{-\tau x^\alpha}_n)| = \infty$$

if $a_j^\pm \neq 0$ for some $j > \alpha + 1/2$. Thus we have $a_j^\pm = 0$ for $j > \alpha + 1/2$.

Next we will show that $a_j^\pm = 0$ for $j > 0$ if $x_0 = 0$. Let $u_1(x)$ be an even function in $D(-\infty, \infty)$ such that $u_1(x) = 1$ for $x$ in $[-1/2, 1/2]$ and $u_1(x) = 0$ for $x$ not in $(-1, 1)$. Put $u_n(x) = u_1(nx)$, $n = 2, 3, \ldots$, and consider the function $u_n(x)e^{-\tau x^\alpha}_n(x)$, $-\infty < x < \infty$. Then we have

$$\Phi_{\pm}(u_ne^{-\tau x^\alpha}_n) \leq |\phi((u_ne^{-\tau x^\alpha}_n)_n) + |\phi((u_ne^{-\tau x^\alpha}_n)p)|.$$

Since $||(u_n)p|| = O(1)$ $(n \to \infty)$ by (1) in the lemma and $||(e^{-\tau x^\alpha}_n)p|| = 1$, we have

$$|\phi((u_ne^{-\tau x^\alpha}_n)p)| \leq ||\phi|| ||(u_ne^{-\tau x^\alpha}_n)p|| \leq ||\phi|| ||(u_n)p|| ||(e^{-\tau x^\alpha}_n)p|| = O(1) \quad (n \to \infty).$$

Moreover, we will claim that $\phi((u_ne^{-\tau x^\alpha}_n)_n) = O(1)$ $(n \to \infty)$. By (1) again, we have

$$||(u_ne^{-\tau x^\alpha}_n)_n|| \leq C \left( \sup_{-1/n \leq x \leq 0} |u_n(x)e^{-2\tau x^\alpha}_n(x)| + n^{-q} \sup_{-1/n \leq x \leq 0} \left| \left( \frac{d}{dx} \right)^q u_n(x)e^{-2\tau x^\alpha}_n(x) \right| \right).$$
We have that
\[
\left( \frac{d}{dx} \right)^j u_n(x)R_n^\alpha(x) = \sum_{k=0}^j jC_k n^{j-k} u_1^{(j-k)}(nx) \times \frac{(-1)^k \Gamma(n+1) \Gamma(\alpha+k+1)}{(\alpha+1)^k \Gamma(n+\alpha+1)} l_{n-k}^{\alpha+k}(x), \quad j = 0, 1, 2, \ldots .
\]

By Perron’s formula in the complex domain (see [7, (8.22.3)]), we have
\[
\sup_{-1/n \leq x \leq 0} \left| \left( \frac{d}{dx} \right)^j u_n(x)R_n^\alpha(x) \right| = O(n^j) \quad (n \to \infty), \quad j = 0, 1, 2, \ldots ,
\]
and thus we have \( \|(u_ne^{-\tau x}R_n^\alpha)_N\| = O(1) \quad (n \to \infty) \). Since
\[
|\phi((u_ne^{-\tau x}R_n^\alpha)_N)| \leq \|\phi\| \|(u_ne^{-\tau x}R_n^\alpha)_N\|,
\]
we have the claim \( \phi((u_ne^{-\tau x}R_n^\alpha)_N) = O(1) \quad (n \to \infty) \). Therefore, we have
\[
\Phi_\pm(u_ne^{-\tau x}R_n^\alpha) = O(1) \quad (n \to \infty).
\]

On the other hand, we have
\[
\Phi_\pm(u_ne^{-\tau x}R_n^\alpha) = \sum_{j=0}^q a_j^\pm \delta^{(j)}(u_ne^{-\tau x}R_n^\alpha)
\]
\[
= \sum_{j=0}^q a_j^\pm (-\tau)^{-j} \sum_{k=0}^j jC_k \frac{n(n-1) \cdots (n-k+1)}{(\alpha+1)^k}.
\]

This implies that, if \( a_j^\pm \neq 0 \) for some \( j > 0 \), then
\[
\lim_{n \to \infty} |\Phi_\pm(u_ne^{-\tau x}R_n^\alpha)| = \infty.
\]

Thus we have that \( a_j^\pm = 0 \) for \( j > 0 \), and therefore we have that \( \phi(f_P) \) is of the form
\[
\phi(f_P) = (\Phi_+(f) + \Phi_-(f))/2
\]
\[
= \begin{cases} 
\sum_{j=0}^p a_j \delta^{(j)}(f), & p \leq \alpha + 1/2 \quad (x_0 > 0), \\
a_0 \delta_{x_0}(f), & (x_0 = 0)
\end{cases}
\]
for \( f \) in \( D(-\infty, \infty) \). From (2) and (3) in the lemma and \( |f(0)| \leq \|f\| \), it follows that \( \phi \) is of the form (\( \ast \)). Q.E.D.

REFERENCES


DEPARTMENT OF MATHEMATICS, COLLEGE OF LIBERAL ARTS, KANAZAWA UNIVERSITY, KANAZAWA 920, JAPAN