

CUT-SET SUMS AND TREE PROCESSES

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ABSTRACT. Suppose that an infinite tree has a value assigned to each vertex. We obtain estimates for the sums of such values over cut-sets of the tree. For certain *tree processes*, where the values are given by random variables, we investigate the almost sure behavior of such cut-set sums. Processes of this type arise in problems concerning random fractals and flows in random networks.

1. Introduction. In this paper, we introduce a type of stochastic process, termed a tree process, which in one sense may be regarded as a generalization of a branching process. Such processes came to the author's attention in connection with problems on statistically self-similar fractals (see [5, 6, 9]), and problems on flows through networks with random edge capacities (see [7]). Tree processes should have diverse applications in modeling situations in which a degree of random replication is present, such as in root or river systems or in genetic phenomena.

Our main aim is to obtain limit theorems for cut-set sums of certain nonnegative tree processes, as are encountered in problems of considerable current interest on finding the almost sure Hausdorff dimension of random fractals and finding flows in randomly capacitated networks. The work described here represents a considerable improvement on earlier estimates and methods.

Roughly speaking, we start with a (directed) tree, with vertices represented by finite sequences of positive integers, with the vertex $\mathbf{i} = i_1, i_2, \dots, i_k$ joined to the vertices $i_1, i_2, \dots, i_k, 1$, $i_1, i_2, \dots, i_k, 2$, etc. Each vertex \mathbf{i} is assigned a value $X_{\mathbf{i}}$ which may be regarded as the capacity of the vertex. By the "min-cut max-flow" theorem a "flow" from the initial vertex to "infinity" through this network is possible only if the infimum of the sums of the capacities over all cut-sets is positive. (A cut-set is a set of vertices that separates the initial vertex from the ones at infinity.) In Theorem 3.1 we obtain sufficient conditions for this to be so.

For the latter part of the paper we take the $X_{\mathbf{i}}$ to be random variables. Our principal results (Theorem 5.1 and its corollaries) are of the form that if the $X_{\mathbf{i}}$ satisfy suitable conditions, for example a martingale type condition $E(\sum_{i=1}^{\infty} X_{\mathbf{i},i} | \mathcal{G}_k) = X_{\mathbf{i}}$, then the infimum value of sums such as $\sum_I |\log X_{\mathbf{i}}|^{\alpha} X_{\mathbf{i}}$ taken over cut-sets I are positive, provided that $\alpha > 1$. This is *not* necessarily the case if the term $|\log X_{\mathbf{i}}|^{\alpha}$ is omitted.

Received by the editors May 26, 1986.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 90B10, 60G48.

Key words and phrases. Tree process, network flow, branching process, random fractal.

This research was carried out whilst the author was a visitor at Oregon State University.

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0002-9939/87 \$1.00 + \$.25 per page

2. Trees. For each nonnegative integer k let $I(k)$ be the set of all k term sequences

$$(2.1) \quad I(k) = \{ \mathbf{i} = i_1, \dots, i_k : i_j \in \mathbb{Z}^+ \}$$

(we make the convention that $I(0)$ contains the null sequence \emptyset). Let $T = \bigcup_{k=0}^{\infty} I(k)$ be the set of all finite sequences. Similarly, let

$$(2.2) \quad A = \{ \mathbf{a} = a_1, a_2, \dots : a_j \in \mathbb{Z}^+ \}$$

be the corresponding infinite sequences. (We observe the convention that sequences \mathbf{i}, \mathbf{j} , etc. are finite and \mathbf{a}, \mathbf{b} , etc. are infinite.)

We let $|\mathbf{i}|$ denote the number of terms in the sequence \mathbf{i} , and we write \mathbf{i}, \mathbf{j} for the sequences obtained by juxtaposition of the terms of \mathbf{i} and \mathbf{j} .

We partially order T by writing $\mathbf{i} \leq \mathbf{j}$ if $\mathbf{j} = \mathbf{i}, \mathbf{q}$ for some sequence \mathbf{q} , that is if \mathbf{j} is formed by augmenting terms to \mathbf{i} . We use similar notation if $\mathbf{i} \in T$ and $\mathbf{a} \in A$. It is natural to regard T as a (directed) tree with vertex \mathbf{i} joined to vertices \mathbf{i}, i for $1 \leq i < \infty$.

If $\mathbf{i} \in T$, we write $\mathbf{i}(r)$ for the curtailment of \mathbf{i} after r terms, so $\mathbf{i}(r) \leq \mathbf{i}$ and $|\mathbf{i}(r)| = r$. If $\mathbf{i}, \mathbf{j} \in T$, let $\mathbf{i} \wedge \mathbf{j}$ denote the maximal sequence \mathbf{q} such that $\mathbf{q} \leq \mathbf{i}$ and $\mathbf{q} \leq \mathbf{j}$.

We term a subset I of T a *cut-set* if for every $\mathbf{a} \in A$ there is a unique sequence $\mathbf{i} \in I$ such that $\mathbf{i} \leq \mathbf{a}$, and if there exists k such that $|\mathbf{i}| \leq k$ for all $\mathbf{i} \in I$. (The latter condition avoids logical difficulties and ensures that the cut-sets are countable.) Intuitively a cut-set separates \emptyset from the "vertices at infinity." Let \mathcal{I} denote the set of all cut-sets of T . There is an induced partial ordering that makes \mathcal{I} into a net: For $I_1, I_2 \in \mathcal{I}$, we write $I_1 \leq I_2$ if for every $\mathbf{i} \in I_2$ there exists $\mathbf{j} \in I_1$ with $\mathbf{j} \leq \mathbf{i}$ (in other words, I_1 separates I_2 from \emptyset). Trivially the sets $I(k)$ are themselves cut-sets with $I(k_1) \leq I(k_2)$ if $k_1 \leq k_2$.

3. Valuations on trees and cut-set sums. Now suppose that a number $X_{\mathbf{i}}$ is associated with each $\mathbf{i} \in T$. For any cut-set I we may form the sum

$$Z_I = \sum_{\mathbf{i} \in I} X_{\mathbf{i}}.$$

For convenience we write Z_k for $Z_{I(k)}$. Our main aim is to investigate the infima and suprema and limiting properties of cut-set sums. We write

$$\liminf_I Z_I = \lim_{k \rightarrow \infty} \inf \{ Z_I : I(k) \leq I \},$$

$$\limsup_I Z_I = \lim_{k \rightarrow \infty} \sup \{ Z_I : I(k) \leq I \}.$$

As usual, we say that $\lim Z_I \rightarrow Z$ (in the net sense) if

$$\liminf_I Z_I = \limsup_I Z_I = Z.$$

For the remainder of the paper we assume that the $X_{\mathbf{i}}$ are nonnegative and decreasing. That is

$$(3.1) \quad X_{\mathbf{i}} \geq 0 \quad \text{if } \mathbf{i} \in T,$$

and

$$(3.2) \quad X_j \leq X_i \quad \text{if } j \geq i.$$

Thus in particular

$$(3.3) \quad \text{if } X_i = 0 \text{ then } X_j = 0 \text{ for all } j \geq i.$$

These assumptions hold in the practical examples encountered so far.

To avoid awkward notation to deal with occurrences of “ $0 \cdot \infty$,” such as in (3.4), we make the convention that sums over subsets of T are taken over those $i \in T$ for which $X_i \neq 0$. Provided that (3.3) holds, this may be done perfectly consistently.

We now prove our basic estimate for infima of cut-set sums in the nonrandom setting. We relate such sums to a type of “energy function” (3.4). In some ways the result is a discrete analogue of the relationship between Hausdorff measures and energy integrals (see for example [4, Theorem 6.4]).

THEOREM 3.1. *Let $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be increasing. Assume that $\{X_i, i \in T\}$ satisfies (3.1)–(3.2). Suppose that for some constant c*

$$(3.4) \quad \sum_{|i|=k} \sum_{|j|=k} \varphi(X_{i \wedge j})^{-1} X_i X_j \leq c$$

for all k , and

$$(3.5) \quad \limsup_{k \rightarrow \infty} \sum_{|i|=k} X_i = m > 0.$$

Then

$$(3.6) \quad \inf_{I \in \mathcal{I}} \sum_{i \in I} \varphi(X_i) > 0.$$

PROOF. Choose $\lambda > 0$ such that $\frac{1}{2}m - c\lambda > 0$. We show that for every cut-set I

$$(3.7) \quad \sum_{i \in I} \varphi(X_i) \geq (\frac{1}{2}m - c\lambda)\lambda > 0.$$

Given a cut-set I we may find k such that $I \leq I(k)$ and $\frac{1}{2}m \leq \sum_{|i|=k} X_i$. Define

$$A = \left\{ i \in I(k) : \lambda \sum_{i(r) \leq j \in I(k)} X_j > \varphi(X_{i(r)}) \text{ for some } r \leq k \right\}.$$

Suppose that $i \in A$. Using (3.2) and that φ is increasing we see that for some $r \leq k$

$$\begin{aligned} \sum_{|j|=k} \varphi(X_{i \wedge j})^{-1} X_j &\geq \sum_{\substack{|j|=k \\ |i \wedge j| \geq r}} \varphi(X_{i \wedge j})^{-1} X_j \\ &\geq \sum_{i(r) \leq j \in I(k)} \varphi(X_{i(r)})^{-1} X_j \geq \lambda^{-1}. \end{aligned}$$

Hence by (3.4)

$$(3.8) \quad \sum_{i \in A} X_i \leq \lambda \sum_{i \in A} \sum_{|j|=k} \varphi(X_{i \wedge j})^{-1} X_i X_j \leq \lambda c.$$

On the other hand, if $r = |\mathbf{j}| \leq k$, we have $\mathbf{j} = \mathbf{i}(r)$ for all $\mathbf{i} \geq \mathbf{j}$, so that

$$(3.9) \quad \begin{aligned} \varphi(X_{\mathbf{j}}) &= \sum_{\substack{|\mathbf{i}|=k \\ \mathbf{i} \geq \mathbf{j}}} \varphi(X_{\mathbf{i}(r)}) \left(\sum_{\substack{|\mathbf{q}|=k \\ \mathbf{q} \geq \mathbf{i}(r)}} X_{\mathbf{q}} \right)^{-1} X_{\mathbf{i}} \\ &\geq \lambda \sum_{\substack{\mathbf{j} \leq \mathbf{i} \in I(k) \\ \mathbf{i} \notin A}} X_{\mathbf{i}}, \end{aligned}$$

provided that $\sum_{|\mathbf{q}|=k, \mathbf{q} \geq \mathbf{i}(r)} X_{\mathbf{q}} > 0$; if not (3.9) is trivial. Thus, since I is a cut-set,

$$\sum_{\mathbf{j} \in I} \varphi(X_{\mathbf{j}}) \geq \lambda \sum_{\substack{|\mathbf{i}|=k \\ \mathbf{i} \notin A}} X_{\mathbf{i}} \geq \lambda \left(\sum_{|\mathbf{i}|=k} X_{\mathbf{i}} - \lambda c \right)$$

by (3.8), giving (3.7). \square

4. Tree processes and examples. We now randomize the valuations on the vertices of T . Let (Ω, \mathcal{F}, p) be a probability space and let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots$ be an increasing sequence of sub- σ -fields of \mathcal{F} . Suppose that for each $\mathbf{i} \in T$ there is a random variable $X_{\mathbf{i}}$. We term $(X_{\mathbf{i}})_{\mathbf{i} \in T}$ a *tree process* with respect to the σ -fields $(\mathcal{F}_k)_{k \in \mathbb{N}}$ provided that $X_{\mathbf{i}}$ is \mathcal{F}_k -measurable for all $\mathbf{i} \in I(k)$.

For the purposes of this paper we shall assume that the tree processes are nonnegative and decreasing, that is (3.1)–(3.3) hold almost surely. To avoid trivial cases we also assume that $0 < E(X_{\emptyset}) < \infty$.

We shall be concerned with the random cut-set sums $(Z_I)_{I \in \mathcal{I}}$ induced by the tree process $(X_{\mathbf{i}})$. Clearly, $\inf_I Z_I, \liminf_I Z_I$, etc. are \mathcal{F} -measurable. We seek conditions on the tree process $(X_{\mathbf{i}})$ that lead to almost sure limiting properties for $(Z_I)_{I \in \mathcal{I}}$.

We term a tree process *independent* if, whenever $|\mathbf{i}| = |\mathbf{j}|$ and $\mathbf{i} \neq \mathbf{j}$, the random variables $X_{\mathbf{i},i}$ and $X_{\mathbf{j},j}$ ($1 < i, j < \infty$) are independent. Note that this allows dependence between $X_{\mathbf{i},i}$ and $X_{\mathbf{i},j}$ as happens in some of the applications.

We now discuss some particular type of tree processes to illustrate the situations that we have in mind and to which the limit theorems of §5 will be applicable.

A. TREE MARTINGALES. It is natural to call a tree process a *tree martingale* L if

$$(4.1) \quad E \left(\sum_{i=1}^{\infty} X_{\mathbf{i},i} \mid \mathcal{F}_{|\mathbf{i}|} \right) = X_{\mathbf{i}} \quad (\mathbf{i} \in T)$$

and a *tree supermartingale* if “=” is replaced by “ \leq ”.

[If $(X_{\mathbf{i}})_T$ is an independent tree (super)martingale, then it is possible to associate a σ -field \mathcal{F}_I with each $I \in \mathcal{I}$ so that $(Z_I, \mathcal{F}_I)_{I \in \mathcal{I}}$ becomes a (super)martingale on the net \mathcal{I} in the generalized sense (see [3, Chapter VI]). Krickeberg [8] obtained almost sure convergence theorems for these martingales if a certain Vitali condition is satisfied. However, such a condition is much too strong for our purposes. Indeed, for the sort of examples that we have in mind, Z_I is not in general a.s. convergent. Moreover, with our definition of independence, this approach becomes notationally cumbersome, and, in any case, we prefer to be a little more general.]

We record the following easy lemma and corollaries.

LEMMA 4.1. *Let (X_i, \mathcal{F}_k) be a tree (super)martingale. Then (Z_k, \mathcal{F}_k) is a (super)martingale.*

PROOF. The measurability condition is clear.

$$Z_{k+1} = \sum_{|i|=k} \sum_{i=1}^{\infty} X_{i,i}$$

so by (4.1)

$$E(Z_{k+1} | \mathcal{F}_k) = \sum_{|i|=k} E\left(\sum_{i=1}^{\infty} X_{i,i} | \mathcal{F}_k\right) = \sum_{|i|=k} X_i = Z_k.$$

For a tree supermartingale equality is replaced by inequality. \square

COROLLARY 4.2. *Let (X_i, \mathcal{F}_k) be a tree (super)martingale. Then for each fixed i*

$$\left(\sum_{|j|=k} X_{i,j}, \mathcal{F}_{|i|+k}\right)_{k \in \mathbb{N}}$$

is a (super)martingale.

PROOF. Apply the lemma to $(X_{ij})_{j \in T}$ regarded as a tree (super)martingale in its own right with respect to the σ -fields $(\mathcal{F}_{k+|i|})_{k \in \mathbb{N}}$. \square

COROLLARY 4.3. *Let (X_i, \mathcal{F}_k) be a nonnegative tree (super)martingale. Then $Z = \lim_{k \rightarrow \infty} Z_k$ exists a.s. with $0 \leq E(Z) \leq E(X_\emptyset)$.*

PROOF. This is immediate by an application of the martingale convergence theorem to (Z_k, \mathcal{F}_k) . \square

In the independent case, a very similar argument shows that the limit $\lim_{r \rightarrow \infty} Z_{I_r}$ exists almost surely for any given increasing sequence of cut-sets I_r .

Thus it might be hoped that, for a tree martingale, $\lim_I Z_I$ exists in the net sense almost surely. Unfortunately, this is not in general the case, as example C below shows (see also [6, Remark 6.12]). Thus the questions of interest revolve around how close we get to net convergence of these sums.

B. BRANCHING PROCESSES. Let N be the progeny distribution of a Galton-Watson branching process. We may regard the branching process as an independent tree process by letting \mathcal{F}_k be the σ -field underlying the first k generations, letting $X_\emptyset = 1$, and for each $i \in I(k)$ with $X_i = 1$ letting

$$X_{i,i} = \begin{cases} 1 & (1 \leq i \leq N), \\ 0 & (N < i) \end{cases}$$

for independent realizations of N . (Of course, $X_{i,i} = 0$ if $X_i = 0$.) Then $(X_i)_T$ is an independent tree process, and if $E(N) = m$ then $(X_i m^{-|i|})_T$ is a tree martingale.

C. SELF-SIMILAR TREE PROCESSES. Processes of this type underlie the Hausdorff dimension calculations related to certain random fractals (see [5, 6, 9]). For this application X_i is the d th power of the diameter of an interval indexed by i in a generalized Cantor set construction.

We term a tree process $(X_i)_T$ *self-similar* if

(a) the r.v. sequences

$$(4.2) \quad (X_{i,1}/X_i, X_{i,2}/X_i, \dots)$$

are independent and identically distributed for each i and independent of $\mathcal{F}_{|i|}$, whenever $X_i \neq 0$,

(b) $X_\emptyset = 1$ a.s.

(Condition (b) is by no means essential; however, it allows us the convenience of all of the sequences (4.2) having the distribution of (X_1, X_2, \dots) .) Thus a self-similar tree process is independent, but of course we do not require that $X_{i,i}$ and $X_{i,j}$ be independent for given i .

Clearly, if $E(\sum_{i=1}^\infty X_i) = 1$ then (X_i) is a tree martingale.

As with branching processes, there is a possibility of “extinction” of self-similar tree processes. Let q be the unique number in $[0, 1]$ satisfying

$$q = \sum_{k=0}^\infty p(\#\{i: X_i > 0\} = k) q^k.$$

Thus q is the extinction probability of the (self-similar) Galton-Watson process given by attaching an individual to the vertices of T for which $X_i > 0$. As usual, if $\sum_{i=1}^\infty X_i > 0$ a.s. then $q = 0$, and if $E(\sum_{i=1}^\infty X_i) > 0$ then $q < 1$.

Part (b) of the next lemma was first proved by Graf [6] in the fractal context.

LEMMA 4.4. *Let (X_i, \mathcal{F}_k) be a self-similar nonnegative tree martingale. Then*

(a) $Z = \lim_{k \rightarrow \infty} Z_k$ exists and is finite a.s. With probability q we have $Z_I = 0$ for all $I \geq I(k)$ for some k , and with probability $1 - q$ we have $Z > 0$.

(b) *Provided that $\sum_{i=1}^\infty X_i$ is not a.s. constant, then*

$$\inf_{I \in \mathcal{I}} Z_I = \liminf_I Z_I = 0,$$

and, with probability $1 - q$,

$$\sup_{I \in \mathcal{I}} Z_I = \limsup_I Z_I = \infty.$$

PROOF. (a) This is established without difficulty by the standard method for treating the extinction probability of a branching process (see [1, §1.5]).

(b) Define the r.v. $W = \inf_{I \in \mathcal{I}} Z_I$. It follows from the self-similarity conditions that the r.v.’s

$$W_i = \inf_{I \in \mathcal{I}} \sum_{i \in I} X_{i,i}/X_i$$

have the distribution of W for each i whenever $X_i > 0$. Then $W = \min\{1, \sum_i X_i W_i\}$ and hence

$$E(W) \leq E\left(\sum_i X_i W_i\right) = \sum_i E(X_i)E(W_i) = E\left(\sum X_i\right)E(W) = E(W).$$

Equality therefore holds, so $W = \sum_i X_i W_i$ a.s. and

$$\text{ess sup } W = \left(\text{ess sup } \sum_i X_i\right)(\text{ess sup } W).$$

Hence either $\text{ess sup } \sum_i X_i = 1$ which would imply that $\sum_i X_i = 1$ a.s., since $E(\sum_i X_i) = 1$, or else $W = 0$ a.s., as required.

It follows that $\sum_{|j|=k} \inf_{i \in I} X_{j,i} = 0$ for all k , so that $\liminf_I Z_I = 0$.

A similar argument, admitting the extra alternative that $E(\inf_{I \in \mathcal{F}} Z_I)$ is infinite, deals with the ‘‘sup’’ case. \square

Many of the standard results on branching processes [1] may be extended without difficulty to nonnegative self-similar tree processes.

D. RECURRENT PROCESSES. This is essentially a generalization of the previous example. It is related to the Hausdorff dimension calculations of the randomizations of the fractals discussed by Bedford [2].

Let s be a positive integer and let S be a subset of $\{1, 2, \dots, s\} \times \{1, 2, \dots, s\}$. We assume that S satisfies a ‘‘transitivity’’ condition in the sense that if $1 \leq i, j \leq s$ there is a sequence $i = i_1, i_2, \dots, i_k = j$ with $(i_r, i_{r+1}) \in S$ for $1 \leq r \leq k - 1$. Let $B = \{\mathbf{i} = i_1, i_2, \dots, i_k \in T: (1, i_1) \in S \text{ and } (i_r, i_{r+1}) \in S \text{ for } 1 \leq r \leq k - 1\}$. Let Y_1, Y_2, \dots, Y_s be given positive r.v.’s. We define a tree process as follows: Take $X_\emptyset = 1$. To start the process, let $X_i = 0$ if $(1, i) \notin S$, and X_i be given by independent realizations of Y_i if $(1, i) \in S$, with \mathcal{F}_1 as the underlying σ -field. Given \mathcal{F}_k , $k \geq 1$, let $X_i = 0$ if $\mathbf{i} \in I(k + 1) \setminus B$. For each $\mathbf{i} = i_1, i_2, \dots, i_k, i_{k+1} \in I(k + 1) \cap B$ take independent realizations of $Y_{i_{k+1}}$ and let $X_{i_1, \dots, i_k, i_{k+1}} = X_{i_1, \dots, i_k} Y_{i_{k+1}}$; this defines \mathcal{F}_{k+1} . Thus (X_i, \mathcal{F}_k) is a tree-process.

Let A be the $s \times s$ matrix with $a_{ij} = E(Y_i)$ if $(i, j) \in S$, $a_{ij} = 0$ otherwise. Then

$$E\left(\sum_{|j|=k} X_{i,j} \mid \mathcal{F}_{|i|}\right) = (1, 1, \dots, 1) A^k \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \begin{matrix} X_i \\ i, \text{th place} \end{matrix}$$

where $\mathbf{i} = i_1, i_2, \dots, i_r$. If $\mathbf{i} = \emptyset$ then the column vector has entry 1 in the first place. In particular

$$E\left(\sum_{|i|=k} X_i\right) = (1, 1, \dots, 1) A^k \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let $\rho > 0$ be the largest absolute value of the eigenvalues of A . If the eigenvalues of maximum modulus all have equal algebraic and geometric multiplicities then there are constants $0 < c_1, c_2 < \infty$ such that

$$(4.3) \quad c_1 X_i \leq \rho^{-k} E\left(\sum_{|j|=k} X_{i,j} \mid \mathcal{F}_{|i|}\right) \leq c_2 X_i$$

for all \mathbf{i} , $\mathcal{F}_{|i|}$, and k . Although $\sum_{|i|=k} X_i$ does not in general converge, one can nevertheless show that, under the conditions given,

$$0 < \liminf_{k \rightarrow \infty} \rho^{-k} \sum_{|i|=k} X_i \leq \limsup_{k \rightarrow \infty} \rho^{-k} \sum_{|i|=k} X_i < \infty$$

almost surely.

5. Almost sure behavior of cut-set sums. We now apply Theorem 3.1 to the situation where the X_i are the random variables of a tree process. As always we assume that the process is nonnegative and decreasing.

THEOREM 5.1. *Let (X_i, \mathcal{F}_k) be an independent tree process such that for some $1 \leq m < \infty$*

$$(5.1) \quad E\left(\sum_{|j|=k} X_{i,j} \mid \mathcal{F}_{|i|}\right) \leq mX_i \quad (k = 1, 2, \dots)$$

and

$$(5.2) \quad E\left(\left(\sum_i X_{i,i}\right)^2 \mid \mathcal{F}_{|i|}\right) \leq mX_i^2$$

for all i and $\mathcal{F}_{|i|}$. Let $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be increasing and suppose that

$$(5.3) \quad \sum_{k=0}^{\infty} E\left(\sum_{|i|=k} \varphi(X_i)^{-1} X_i^2\right) < \infty.$$

Then $\inf_{I \in \mathcal{F}} \sum_{i \in I} \varphi(X_i) > 0$ a.s. whenever

$$\limsup_{k \rightarrow \infty} \sum_{|i|=k} X_i > 0.$$

PROOF. We estimate the expectation of the expression (3.4) to enable us to apply Theorem 3.1. First observe that if $|q| = q$ and $s \geq 0$ then

$$\begin{aligned} & E\left(\sum_{i \neq j} \sum \left(\left(\sum_{|i'|=s} X_{q,i,i'}\right)\left(\sum_{|j'|=s} X_{q,j,j'}\right)\right) \mid \mathcal{F}_{q+1}\right) \\ &= \sum_{i \neq j} \sum E\left(\sum_{|i'|=s} X_{q,i,i'} \mid \mathcal{F}_{q+1}\right) E\left(\sum_{|j'|=s} X_{q,j,j'} \mid \mathcal{F}_{q+1}\right) \\ &\leq m^2 \sum_{i \neq j} \sum X_{q,i} X_{q,j} \end{aligned}$$

by (5.1). Hence

$$(5.4) \quad \begin{aligned} & E\left(\sum_{i \neq j} \sum \left(\left(\sum_{|i'|=s} X_{q,i,i'}\right)\left(\sum_{|j'|=s} X_{q,j,j'}\right)\right) \mid \mathcal{F}_q\right) \\ &\leq m^2 E\left(\left(\sum_i X_{q,i}\right)^2 \mid \mathcal{F}_q\right) \leq m^3 X_q^2 \end{aligned}$$

by (5.2). Since

$$(5.5) \quad \begin{aligned} & \sum_{|i|=k} \sum_{|j|=k} \varphi(X_{i \wedge j})^{-1} X_i X_j \\ &\leq \sum_{q=0}^{k-1} \sum_{|q|=q} \varphi(X_q)^{-1} \sum_{i \neq j} \sum \left(\sum_{|i'|=k-q-1} X_{q,i,i'}\right) \left(\sum_{|j'|=k-q-1} X_{q,j,j'}\right) \\ &\quad + \sum_{|q|=k} \varphi(X_q)^{-1} X_q^2 \end{aligned}$$

it follows from (5.2) and (5.4) that (5.5) has unconditional expectation bounded by $m^3 \sum_{q=0}^k E(\sum_{|i|=q} \varphi(X_i)^{-1} X_i^2)$. Hence (5.5) is almost surely bounded in k , by (5.3), and the conclusion follows by Theorem 3.1. \square

The interpretation of condition (5.3) becomes clearer in the following corollary.

COROLLARY 5.2. *Let (X_i, \mathcal{F}_k) be an independent tree process such that (5.1) and (5.2) hold. Suppose that for some $0 < \gamma < 1$ we always have*

$$(5.6) \quad X_{i,i} \leq \gamma X_i.$$

Suppose that $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a decreasing function such that $t \mapsto t\psi(t)$ is increasing and

$$(5.7) \quad \int_0^\infty \frac{dt}{t\psi(t)} < \infty.$$

Then

$$(5.8) \quad \inf_{I \in \mathcal{I}} \sum_{i \in I} \psi(X_i) X_i > 0$$

and

$$(5.9) \quad \liminf_I \sum_{i \in I} \psi(X_i) X_i = \infty$$

a.s. whenever

$$\limsup_{k \rightarrow \infty} \sum_{|i|=k} X_i > 0,$$

and further, a.s.

$$(5.10) \quad \limsup_I \sum_{i \in I} \psi(X_i)^{-1} X_i = 0.$$

PROOF. Let $\varphi(t) = t\psi(t)$. Then φ is increasing, as is the function $t \mapsto \varphi(t)^{-1}t$, so by (5.6)

$$\begin{aligned} \sum_{k=0}^\infty E\left(\sum_{|i|=k} \varphi(X_i)^{-1} X_i^2\right) &\leq \sum_{k=0}^\infty \varphi(\gamma^k)^{-1} \gamma^k E\left(\sum_{|i|=k} X_i\right) \\ &\leq \sum_{k=0}^\infty \varphi(\gamma^k)^{-1} \gamma^k m E(X_\phi) \end{aligned}$$

by (5.1). But this series converges using (5.7), and so (5.3) holds. Thus (5.8) follows from Theorem 5.1. To get (5.9) we apply this result using a function ψ_1 , chosen so that $\lim_{t \rightarrow 0} \psi_1(t)/\psi(t) = \infty$ but so the other hypotheses remain true for ψ_1 .

Finally, (5.10) follows from (5.3), noting that

$$\sup_{I \geq I(k)} \sum_{i \in I} \psi(X_i)^{-1} X_i \leq \sum_{q=k}^\infty \sum_{|i|=q} \psi(X_i)^{-1} X_i. \quad \square$$

The conditions for ψ in the corollary are satisfied by functions such as $(\log \frac{1}{t})^\alpha$, $\log \frac{1}{t} (\log \log \frac{1}{t})^\alpha$, etc. provided that $\alpha > 1$.

Theorem 5.1 and Corollary 5.2 are applicable to many of the situations discussed in §4. For example, (5.1) is rather weaker than the (super)martingale condition (see Corollary 4.2), and also holds for the recurrent processes where $\rho \leq 1$ (4.3). Condition (5.2), which controls the variances of $\sum_i X_{i,i}$, is automatically satisfied for self-similar tree processes if $E((\sum_i X_i)^2) < \infty$, with a similar situation for recurrent processes. Two particular cases are covered by the following corollaries.

COROLLARY 5.3. *Let (X_i, \mathcal{F}_k) be a self-similar independent tree martingale such that $E((\sum_i X_i)^2) < \infty$ and always $0 \leq X_i < \gamma < 1$. Let ψ be as in Corollary 5.2. Then (5.10) holds a.s., and a.s. either $X_i = 0$ for all i with $|i| \geq k$ for some k , or (5.8) and (5.9) hold.*

COROLLARY 5.4. *Let (X_i, \mathcal{F}_k) be an independent recurrent process (as in §4D) with $\rho = 1$. Suppose that $0 < Y_i < \gamma < 1$. Then if ψ is as in Corollary 5.2, (5.8)–(5.10) hold a.s.*

A consequence of Corollary 5.3 is that the statistically self-similar fractals of the form shown in [5, 6, 9] to have almost sure Hausdorff dimension d , must almost surely have infinite Hausdorff measure with respect to the measure functions $h(t) = t^d (\log \frac{1}{t})^\alpha$, $h(t) = t^d \log \frac{1}{t} (\log \log \frac{1}{t})^\alpha$, etc. for any $\alpha > 1$.

6. Concluding remarks. Clearly this paper raises many more questions than it answers. How far can the conditions in Theorem 5.1 and its corollaries be relaxed? What are the exact critical functions ψ for the lower and upper limits of the cut-set sums to be positive and finite? To what extent can the independence conditions be weakened?—certainly to some extent, but not completely. What about the limit behavior of tree processes allowed to take both positive and negative values? Can the ideas be extended to continuous parameter tree processes?

REFERENCES

1. K. B. Athreya and P. E. Ney, *Branching processes*, Springer-Verlag, Berlin and New York, 1972.
2. T. Bedford, *Dimension and dynamics for fractal recurrent sets*, J. London Math. Soc. **33** (1986), 89–100.
3. L. Egghe, *Stopping time techniques for analysts and probabilists*, Cambridge Univ. Press, 1984.
4. K. Falconer, *The geometry of fractal sets*, Cambridge Univ. Press, 1985.
5. ———, *Random fractals*, Math. Proc. Cambridge Philos. Soc. **100** (1986), 559–582.
6. S. Graf, *Statistically self-similar fractals*, Prob. Theory Related Fields **74** (1987), 357–392.
7. G. R. Grimmett, *Random flows: network flows and electrical flows through random media*, Surveys in Combinatorics, Cambridge Univ. Press, 1985, pp. 59–95.
8. K. Krickeberg, *Convergence of martingales with a directed index set*, Trans. Amer. Math. Soc. **83** (1956), 313–337.
9. R. D. Mauldin and S. C. Williams, *Random constructions, asymptotic geometric and topological properties*, Trans. Amer. Math. Soc. **295** (1986), 325–346.

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