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BRANCHED COVERS AND CONTACT STRUCTURES

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ABSTRACT. The object of this paper is to prove the following

THEOREM. *Every closed, orientable three-manifold has a parallelization by three contact forms.*

1. Introduction. Throughout this paper, M stands for a closed, oriented three-manifold.

DEFINITION. A 1-form ω on an odd-dimensional manifold is called a contact form if $\omega \wedge d\omega \wedge \cdots \wedge d\omega$ is a volume form.

REMARK. For any function f and any 1-form ω, we have the identity $f\omega \wedge (d(f\omega))^n = f^{n+1}\omega \wedge (d\omega)^n$. So if f has no zeros then $f\omega$ is a contact form if and only if ω is a contact form. This means that the property of being a contact form is determined by the hyperplane distribution of the kernels of ω at each point.

DEFINITION. A contact structure is a hyperplane distribution which is locally the kernel of contact forms, i.e. every point has a neighborhood and a contact form on that neighborhood which annihilates the hyperplanes of the distribution.

DEFINITION. Let N_1, N_2 be manifolds of the same odd dimension, with contact forms ω_1, ω_2 respectively. Then ω_1 is equivalent to ω_2 if there is a diffeomorphism $\Phi\colon N_1 \to N_2$ and there is a nowhere zero function f on N_1 such that the equality $\Phi^*\omega_2 = f\omega_1$ is satisfied. Equivalently, the diffeomorphism takes the hyperplane distribution $\operatorname{Ker}\omega_1$ into $\operatorname{Ker}\omega_2$.

In $\mathbf{R}^3$, with coordinates x, y, z, the standard contact form is $x\,dy + dz$.

Let $S^3 \hookrightarrow \mathbf{R}^4$ be the usual inclusion, with component functions x_1, x_2, x_3, x_4; then the standard contact form on the sphere S^3 is $x_1dx_2 - x_2dx_1 + x_3dx_4 - x_4dx_3$.

DEFINITION. Let N be a manifold of dimension $2n+1$. An almost contact structure on N is a pair (ω, Ω), where ω is a 1-form on N and Ω is a 2-form on N, such that $\omega \wedge \Omega^n$ is a volume form.

The space of almost contact structures on N is homotopy equivalent to the space of reductions of the structure group of TN to $U(n)$, acting on $\mathbf{R}^{2n+1}$ in the obvious way determined by the factorization $\mathbf{R}^{2n+1} = \mathbf{R} \oplus \mathbf{C}^n$.

In 1966 Chern [3] posed the question of the existence of a contact form on $S^1 \times S^2 \# \mathbf{R}P^3$.

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In 1969 Gromov [4] proved that for *open* odd-dimensional manifolds the inclusion of the space of contact forms into that of almost contact structures is a weak homotopy equivalence.

Martinet [9] gave the first affirmative answer for closed manifolds: he proved that every closed, orientable manifold of dimension 3 has a contact form. He used methods from Lutz [7] and a surgery description [6, 12] of M.

Some time later, Thurston and Winkelnkemper [11] gave another proof using an open book decomposition [1, 13] of M.

In addition to the problem of existence of a contact form, there is the problem of existence of several nonequivalent contact forms, in the sense of the previous definition. Note that it is not the same as being *isomorphic as forms*, in which case we have an equality $\Phi^*\omega_2 = \omega_1$.

In 1982 Bennequin [2] showed that a certain contact form on $\mathbf{R}^3$ is not equivalent to the standard one. He also found a contact form on S^3 which is nonequivalent to the standard form, yet homotopic to it through nowhere zero 1-forms.

There remains the question, for a *closed* odd-dimensional manifold, of the existence of a contact form in each homotopy class of almost contact structures.

In this paper, we give a short proof of the existence of three everywhere independent contact forms on M.

REMARK. It is also our purpose to point out that a not everywhere smooth map can sometimes lift a differential form.

We use as a main tool the theorem of Hilden, Montesinos, and Thickstun [5] that describes M as a 3-fold simple branched cover of S^3, such that the covering map $M \to S^3$ fails to be a local diffeomorphism only along a curve C which is the boundary of a nonsingular disk.

Our construction is as follows: Let ω be the standard contact form on S^3 and let $h\colon M \to S^3$ be such a branched cover. It is possible to modify ω and h slightly to get a contact form ω' and a branched cover H that is C^∞ except at the points of C, and such that $\omega_1 = H^*\omega' \in \Lambda^1(M - C)$ extends, just by continuity, to a C^∞ contact form on all of M. Also, the plane distribution $\operatorname{Ker}\omega_1$ is a trivial 2-plane bundle on M and we only have to apply the following:

PROPOSITION 1. *For a contact form ω_1 on M, the following are equivalent*:

(1) *There are contact forms ω_2, ω_3 with $\omega_1 \wedge \omega_2 \wedge \omega_3$ nowhere zero.*

(2) *There are arbitrary 1-forms α, β with $\omega_1 \wedge \alpha \wedge \beta$ nowhere zero.*

(3) *The 2-plane bundle $\operatorname{Ker}\omega_1$ is trivial.*

The classical example of a contact form is the Liouville-Cartan form. If N is a surface with a Riemann metric and if $\pi\colon S^*N \to N$ is its cotangent sphere bundle, define a 1-form ξ on S^*N by

$$\xi_{v^*} = v^* \circ \pi_* \quad \text{for all } v^* \in S^*N.$$

Then ξ is called the Liouville-Cartan form of S^*N and it is always a contact form.

Now assume N is oriented, and let $\rho\colon FN \to N$ be the bundle of oriented orthonormal frames on N. This defines an almost complex structure J on N by the conditions

$\|Jv\| = \|v\|$,

if $v \neq 0$, then $\{v, Jv\}$ is an oriented orthogonal frame.

Consider the bundle equivalences Φ, Ψ: $S^*N \to FN$, given by

$$\Phi(v^*) = \{v, Jv\}, \quad \Psi(v^*) = \{-Jv, v\}, \qquad v^* = \langle v, \cdot \rangle,$$

and consider the canonical 1-forms θ^1 and θ^2 on FN. Then

$$\Phi^*\theta^1 = \xi, \qquad \Psi^*\theta^2 = \xi;$$

thus θ^1 and θ^2 are isomorphic to the Liouville-Cartan form, therefore contact forms.

If η is the connection form and if Ω is the curvature form, then $\eta \wedge d\eta = \eta \wedge \Omega$ and $\eta \wedge \theta^1 \wedge \theta^2$ is nowhere zero. Now if N has genus not equal to 1 then it posesses a metric with nowhere zero curvature and η is a contact form. If N is a torus, then take $\omega = \eta + (\rho^*f)\theta^1 + (\rho^*g)\theta^2$, with f, $g \in C^\infty(N)$; now $\omega \wedge \theta^1 \wedge \theta^2$ has no zeros and it is shown in **[8]** that f and g can be so chosen that ω is a contact form.

In summary, FN is the classical example of the theorem we prove in this paper. Indeed, FN has specific relations:

$$d\theta^1 = -\eta \wedge \theta^2, \qquad d\theta^2 = \eta \wedge \theta^1.$$

The parallelization we shall obtain for general M does not necessarily satisfy these structure equations.

2. Proof of the theorem.

DEFINITION. An n-fold branched cover of a 3-manifold M_1 by another one M is a map h: $M \to M_1$ such that there are links $L \subset M$, $L_1 \subset M_1$ satisfying

(i) $L = h^{-1}(L_1)$.

(ii) The restriction h: $M - L \to M_1 - L_1$ is an ordinary n-fold covering map.

We say that h is branched over L_1.

One can assume further that there are tubular neighborhoods $U \supset L$ and $U_1 \supset L_1$, with polar coordinates $(r_0, \theta_0, \varphi_0)$ and $(r_1, \theta_1, \varphi_1)$ respectively, where θ corresponds to longitude and φ to meridian, such that the expression of h in these coordinates is

$$h(r_0, \theta_0, \varphi_0) = (r_0, l\theta_0, m\varphi_0 + k\theta_0)$$

for some integers l, m, k, with l, $m > 0$.

DEFINITION. Let h: $M \to M_1$ be an n-fold cover, branched over $L_1 \subset M_1$. Given a point $P \in M_1 - L_1$, there is a canonical action of the group $\pi_1(M_1 - L_1, P)$ on the set of n elements represented by $h^{-1}(P)$, thus giving a representation of the group $\pi_1(M_1 - L_1, P)$ into the symmetric group of n letters. The branched cover is called *simple* if every meridian of L_1 is represented by a transposition.

Our main tool will be the following **[5, 10]**:

BRANCHED COVER THEOREM. *There exists a 3-fold simple cover h: $M \to S^3$, branched over a knot $K \subset S^3$, such that the set C of points where h is not a local diffeomorphism bounds in M an embedded locally flat disk. Thus there is a smoothly embedded 3-ball $B \subset M$ with $C \subset \mathring{B}$.*

Let $L = h^{-1}(K)$, link whose components will be denoted by C_0, $C_1, \dots, C_s$, and $C_0 = C$ above. Consider a local expression of h around a component C_i:

$$h(r_0, \theta_0, \varphi_0) = (r_0, l\theta_0, m\varphi_0 + k\theta_0).$$

If $i = 1, 2, \dots, s$, then h is a local diffeomorphism at the points of C_i and so $m = 1$. If $i = 0$, we must have $m > 1$ so h is not a local diffeomorphism, also $lm \leqslant 3$ and the possibilities are $m = 2$ or $m = 3$, with $l = 1$ in either case. It is easy to see that for $m = 3$ a meridian of K in S^3 induces a permutation which is a cycle of length three, thus it is an even permutation and it cannot be conjugate to a transposition, in contradiction with h being simple.

As a consequence, we have $m = 2$. We also conclude that $s = 1$.

Thus L has two components: $L = C \cup C_1$, h is a diffeomorphism in a whole neighborhood of C_1 and its local expression around C is

$$h(r_0, \theta_0, \varphi_0) = (r_0, \theta_0, 2\varphi_0 + k\theta_0).$$

Let $\omega = x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3$ be the standard contact form on S^3.

Let $\Gamma = \{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 = 1\}$; this is a circle transverse to ω, i.e. $\omega(\dot{\Gamma}) > 0$. By **[1]**, we can assume that K is a braid contained in a tubular neighborhood W of Γ.

Now shrink W towards Γ by an ambient isotopy. This takes K to a braid K' whose tangent lines are close to those of Γ. So we can also assume that K is transverse to ω.

Our next tool is the following result [**9**, p. 151]:

PROPOSITION 2 (MARTINET). *Let K be a simple closed curve transverse to a contact form ω. There is a tubular neighborhood of K, with coordinates $(\theta, x_1, \dots, x_n, y_1, \dots, y_n)$, where θ is the angular coordinate and x_i, y_i vanish along K, and there is a nonvanishing function f on this neighborhood such that*

$$f\omega = d\theta + \sum_{i=1}^{n} (x_i dy_i - y_i dx_i) \quad \textit{and} \quad \dot{K} = \left.\frac{\partial}{\partial\theta}\right|_K.$$

A brief summary of Martinet's argument is the following: First, find a function g such that along K we have the equalities $g\omega(\dot{K}) = 1$ and $\dot{K} \lrcorner d(g\omega) = 0$.

Second, let U be a tube around K thin enough so there is a diffeomorphism Ψ: $U \to S^1 \times \mathbf{R}^{2n}$ taking $\dot{K}$ to $\dot{K}_0$, where $K_0 = S^1 \times 0$, and satisfying $\Psi^*(\Sigma dx_i \wedge dy_i) = d\omega$ along K. This is possible because the symplectic group $\mathrm{SP}(2n, \mathbf{R})$ is connected. Now, the forms

$$\omega_0 = \Psi^*\big(d\theta + \textstyle\sum (x_i dy_i - y_i dx_i)\big), \qquad \omega_1 = g\omega$$

satisfy $\omega_0 = \omega_1$ and $d\omega_0 = d\omega_1$ along K.

Third, consider the family $\omega_t = (1 - t)\omega_0 + t\omega_1$ and apply to it the proof of Gray's stability theorem, as it appears in [**8**, p. 1], or equivalently in [**9**]. We get an isotopy Φ_t such that

$$\Phi_t(P) = P \quad \text{for all } t \text{ and for } P \in K,$$

$$\Phi_t^* \omega_0 = f_t \omega_t \quad \text{for some function } f_t.$$

Then

$$\begin{aligned}(f_1 g)\omega &= f_1\omega_1 = \Phi_1^*\omega_0 \\ &= (\Psi\Phi_1)^*\left(d\theta + \sum (x_i dy_i - y_i dx_i)\right)\end{aligned}$$

and $\Psi\Phi_1$ provides the desired coordinates. This proves Martinet's result.

In our case, we have a branched cover $h\colon M \to S^3$ which is a local diffeomorphism except at the points of the circle $C \subset M$. The knot $K \subset S^3$, over which h is branched, has a coordinate tube $(U_1, (\theta, x, y))$ where $f\omega = d\theta + x dy - y dx$ for some function f. Extend f to a positive function on all of S^3, and define $\omega' = f\omega$. If we denote by (r, θ, φ) the polar coordinates corresponding to (θ, x, y), then

$$\omega' = d\theta + r^2 d\varphi \quad \text{in } U_1 - K.$$

For U_1 thin enough, we find polar coordinates $(r_1, \theta_1, \varphi_1)$ on U_1 and polar coordinates $(r_0, \theta_0, \varphi_0)$ on a tube U around C, such that the local expression of h is

$$h(r_0, \theta_0, \varphi_0) = (r_0, \theta_0, 2\varphi_0 + k\theta_0).$$

Now the coordinate systems (r, θ, φ) and $(r_1, \theta_1, \varphi_1)$ on U_1 can be supposed to satisfy $\theta = \theta_1$ along K, and they define isotopic diffeomorphisms of U_1 onto $S^1 \times B^2$, rel K, if and only if they determine equivalent framings of K. This means that for some integer p the coordinate systems on U_1,

$$(r, \theta, \varphi) \quad \text{and} \quad (r_1, \theta_1, \varphi_1 + p\theta_1),$$

are isotopic by an isotopy preserving the values of θ along K and also preserving the condition $r = 0$ along K.

In the coordinates $(r_0, \theta_0, \varphi_0)$ and $(r_1, \theta_1, \varphi_1 + p\theta_1)$, the local expression of h is

$$h(r_0, \theta_0, \varphi_0) = (r_0, \theta_0, 2\varphi_0 + (k + p)\theta_0).$$

Because of these considerations, it is possible to construct a global isotopy Φ_t of S^3, supported in U_1, fixing the points of K, and with $\Phi_0 = \mathrm{Id}$, such that the branched cover $H = \Phi_1 \circ h$ has the following local expression in the coordinates $(r_0, \theta_0, \varphi_0)$ and (r, θ, φ), in smaller neighborhoods also denoted U, U_1:

$$r = r_0, \quad \theta = \theta_0, \quad \varphi = 2\varphi_0 + (k + p)\theta_0.$$

On $M - C$, the map H is a local diffeomorphism, so the form defined as $\omega_1 = H^*\omega'$ is a contact form on $M - C$. Its local expression in the coordinates $(r_0, \theta_0, \varphi_0)$ is

$$\begin{aligned}\omega_1 &= d\theta_0 + r_0^2 d(\varphi_0 + (k + p)\theta_0) \\ &= \left(1 + (k + p) r_0^2\right) d\theta_0 + r_0^2 d\varphi_0, \quad \text{on } U - C.\end{aligned}$$

It follows that ω_1 extends to all of U as a C^∞ contact form. We now have a contact form on all of M which we also denote ω_1.

REMARK. If we try to use the smooth version of the branched cover,

$$r = r_0^2, \quad \theta = \theta_0, \quad \varphi = 2\varphi_0 + (k + p)\theta_0,$$

then the resulting form is not a contact form at the points of C. Thus it is the nonsmooth version of H, the one with $r = r_0$, the convenient one for our purpose.

We will prove now that $\operatorname{Ker}\omega_1$ is a trivial 2-plane bundle. First note that $\operatorname{Ker}\omega'$ is trivial, because 2-plane bundles over S^3 are classified by the homotopy group $\pi_2(S^1) = 0$. Now take a 3-ball $B \subset M$ with $C \subset \mathring{B}$. Then H is a local diffeomorphism on $M - \mathring{B}$, thus $(\operatorname{Ker}\omega_1)|_{M-\mathring{B}}$ is trivial since $\omega_1 = H^*\omega'$. Finally $[\partial B, S^1] = 0$ implies $\operatorname{Ker}\omega_1$ is trivial.

In order to finish the proof of the theorem, we only have to prove Proposition 1.

3. Proof of Proposition 1. That (2) and (3) are equivalent is a standard fact. That (1) implies (2) is obvious. That (2) implies (1) follows from the fact that for any number $\delta > 0$ the forms $\omega_1, \omega_1 + \delta\alpha, \omega_1 + \delta\beta$ are everywhere independent, and for small δ they are C^1-close to ω_1, hence contact forms.

Another way to see this is to take two everywhere independent vector fields X, $Y \in \operatorname{Ker}\omega_1$ and to realize that because the product $\omega_1 \wedge d\omega_1$ is nowhere zero the same is true for the product $\omega_1 \wedge L_X\omega_1 \wedge L_Y\omega_1$. Thus if F_t is the flow of X and if G_t is the flow of Y then for small t the differential form $\omega_1 \wedge F_t^*\omega_1 \wedge G_t^*\omega_1$ has no zeros. Being isomorphic to ω_1, the forms $F_t^*\omega_1$ and $G_t^*\omega_1$ are contact forms.

PROPOSITION 3. *Giving M a contact form ω with* $\operatorname{Ker}\omega$ *trivial is equivalent to giving M a parallelization by vector fields X, Y, Z such that $[X, Y] = Z$.*

In one direction, if X, $Y \in \operatorname{Ker}\omega$ are everywhere independent then $\omega \wedge d\omega$ nowhere zero implies that $Z = [X, Y]$ is nowhere tangent to $\operatorname{Ker}\omega$, thus independent of X and Y.

In the other direction, if there are vector fields X, Y such that X, Y, and $[X, Y]$ are everywhere independent, then the form ω given by

$$\omega(Z) = 1, \qquad \omega(X) = \omega(Y) = 0$$

is a contact form and $\operatorname{Ker}\omega$ is trivial.

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