PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 2, October 1987

UNIQUENESS OF THE DYER-LASHOF OPERATIONS

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ABSTRACT. It is well known that the Steenrod operations are uniquely determined by certain of their properties, namely stability, the Cartan formula, and the unstable property. In this note we give a similar characterization of Dyer-Lashof operations.

The Dyer-Lashof operations Q^i are natural homomorphisms in the mod p homology of E_∞ spaces which have the following properties (see [**2**] and [**1**, I §1]; we remind the reader that E_∞ spaces need not be connected).

(A) If $p = 2$, Q^i has degree i and $Q^i x = 0$ (respectively, $Q^i x = x^2$) if the degree of x is greater than i (respectively, equal to i). If p is odd, Q^i has degree $2i(p-1)$ and $Q^i x = 0$ ($Q^i x = x^p$) if the degree of x is greater than $2i$ (equal to $2i$).

(B) The external Cartan formula holds:

$$Q^i(x \times y) = \sum_{j \geqslant 0} Q^j x \times Q^{i-j} y.$$

(C) $Q^i \sigma = \sigma Q^i$, where $\sigma\colon \tilde{H}_n \Omega X \to H_{n+1} X$ is the homology suspension.

Our main result is the following.

THEOREM. *Let R^i, $i \geqslant 0$, be any sequence of natural homomorphisms in the homology of E_∞ spaces satisfying* (A), (B) *and* (C). *Then $R^i = Q^i$.*

This is analogous to the uniqueness theorem for Steenrod operations given in [**5**, Chapter VIII] and [**3**, §10], but the proof will be quite different.

The proof will consist of several lemmas. To simplify the notation we begin with the case $p = 2$, so that H_* denotes mod 2 homology. The odd primary case goes through with some minor changes which will be noted at the end.

Let $E_j X$ denote $E\Sigma_j \times_{\Sigma_j} X^j$, where $E\Sigma_j$ is a contractible free Σ_j-space and the symmetric group operators on X^j by permuting the factors. Let $i\colon X \times X \to E_2 X$ be the inclusion, and let

$$d_j\colon E_j(X \times Y) \to E_j X \times E_j Y$$

be the map taking $(e, x_1, y_1, \ldots, x_j, y_j)$ to $(e, x_1, \ldots, x_j, e, y_1, \ldots, y_j)$. Recall that if x is an element of $H_n X$ then $Q^i x$ is defined to be $(\xi_2)_*(e_{i-n} \otimes x^2)$, where the map $\xi_2\colon E_2 X \to X$ is part of the E_∞ structure for X. We can give a similar factorization for R^i by using the construction CX given in [**4**, §2]. The space $C(X^+)$ is

Received by the editors April 17, 1984.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55S12.

homeomorphic to $(\coprod_{j\geqslant 1} E_jX)^+$, and we write $\pi_j\colon C(X^+)\to E_jX$ for the projection and $\lambda\colon X\to X^+$ for the inclusion. For each $j\geqslant 1$ and each (not necessarily E_∞) space X we let $\overline{R}{}^i_j$ be the composite

$$H_nX\xrightarrow{\lambda_*}H_n(X^+)\xrightarrow{\eta_*}H_nC(X^+)\xrightarrow{R^i}H_{n+i}C(X^+)\xrightarrow{\pi_j}H_{n+i}E_jX;$$

note that if R^i is equal to Q^i we must have $\overline{R}{}^i_j=0$ for $j\neq 2$ and $\overline{R}{}^i_2x=e_{i-n}\otimes x^2$ for $x\in H_nX$.

Lemma 1. *If $x\in H_nX$ then*

$$\overline{R}{}^n_jx=\begin{cases} i_*(x\times x) & \text{if } j=2,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The diagram

$$\begin{array}{ccc} X\times X & \xrightarrow{i} & E_2X\\ \downarrow{\scriptstyle(\eta\circ\lambda)\times(\eta\circ\lambda)} & & \downarrow\\ C(X^+)\times C(X^+) & \rightarrow & C(X^+)\end{array}$$

where the bottom map is the product map in $C(X^+)$, commutes by the definition of the product. The result follows from property (A).

Lemma 2. $(d_j)_*\overline{R}{}^i_j(x\times y)=\sum_{k\geqslant 0}\overline{R}{}^k_jx\times\overline{R}{}^{i-k}_jy.$

Proof. Let $d\colon C((X\times Y)^+)\to C(X^+)\times C(Y^+)$ be induced by the projections p_1 and p_2 of $(X\times Y)^+$ onto X^+ and Y^+. Then d is an E_∞ map, and hence

$$\begin{aligned} d_*R^i\eta_*\lambda_*(x\times y) &= R^id_*\eta_*\lambda_*(x\times y)\\ &= R^i(\eta\times\eta)_*(p_1\times p_2)_*\lambda_*(x\times y)\\ &= R^i(\eta_*\lambda_*x\times\eta_*\lambda_*y)\\ &= \sum_{k\geqslant 0}R^k\eta_*\lambda_*x\times R^{i-k}\eta_*\lambda_*y\end{aligned}$$

by property (B). The result follows since $(\pi_j\times\pi_j)\circ d=d_j\circ\pi_j$.

Lemma 3. $\overline{R}{}^i_j=0$ *for* $j\neq 2$.

Proof. Let P be the one-point space and let $g\colon E_jP\to P$ be the unique map. Let $e\in H_0P$ be the identity element. The composite

$$E_jX=E_j(X\times P)\xrightarrow{d_j}E_jX\times E_jP\xrightarrow{1\times g}E_jX\times P=E_jX$$

is the identity. Hence for any $x\in H_nX$ we have

$$\overline{R}{}^i_jx=(1\times g)_*(d_j)_*\overline{R}{}^i_j(x\times e)=\sum_{k\geqslant 0}\overline{R}{}^{i-k}_jx\times g_*\overline{R}{}^k_je$$

by Lemma 2. Now $g_*\overline{R}{}^k_je=0$ for $k>0$ for dimensional reasons, and $\overline{R}{}^0_je=0$ if $j\neq 2$ by Lemma 1. The result follows.

LEMMA 4. *If X is an E_∞ space and $x \in H_n X$ then $R^i x = (\xi_2)_* \overline{R}^i_2 x$.*

PROOF. Consider the diagram

$$\begin{tikzcd}
H_nX \arrow[rr, "\overline{R}^i_2"] \arrow[d, "\lambda_*"] \arrow[dd, bend right=60, equal] & & H_{n+i}E_2X \arrow[d, hook] \arrow[dr, "(\xi_2)_*"] & \\
H_nX^+ \arrow[r, "\eta_*"] \arrow[d] & H_nC(X^+) \arrow[r, "R^i"] \arrow[d] & H_{n+i}C(X^+) \arrow[d] & H_{n+i}X \\
H_nX \arrow[r, "\eta_*"] & H_nCX \arrow[r, "R^i"] & H_{n+i}CX \arrow[ur, "\xi_*"'] &
\end{tikzcd}$$

Here ξ is the map given by the E_∞ structure and the triangle commutes by definition of ξ and ξ_2. The upper rectangle commutes by Lemma 3. The two lower rectangles (in which the unmarked arrows are induced by the projection $X^+ \to X$) commute by naturality. Since ξ is an E_∞ map we have $\xi_* R^i \eta_* x = R^i \xi_* \eta_* x = R^i x$ as required.

By Lemma 4 it suffices to show $\overline{R}^i_2 = \overline{Q}^i_2$ for all i, where $\overline{Q}^i_2$ is constructed from Q^i in the same way that $\overline{R}^i_2$ is constructed from R^i.

As usual, we filter CX by letting $F_k CX$ be the image of

$$\Big(\coprod_{1 \leqslant j \leqslant k} E_j X \Big)^+ \subset C(X^+) \to CX.$$

We write $F_k H_* CX$ for the image of the map $H_* F_k CX \to H_* CX$; recall from the proof of [**1**, Theorem I.4.1] that this map is in fact a monomorphism. By Lemma 3 we see that $R^i \eta_* x$ is in $F_2 H_* CX$ for any $x \in H_n X$.

Now let $s_n \in \tilde{H}_n S^n$ be the fundamental class.

LEMMA 5. *$R^i \eta_* s_n = Q^i \eta_* s_n$ in $H_{n+1} C(S^n)$ for all i and n.*

PROOF. The result is immediate from property (A) if $i \leqslant n$, so we assume $i > n$. The proof of [**1**, Theorem I.4.1] shows that $F_2 H_{n+i} C(S^n)$ is generated by $Q^i \eta_* s_n$, hence $R^i \eta_* s_n$ is a scalar multiple of $Q^i \eta_* s_n$. To determine what multiple we consider the composite

$$F_2 H_{n+i} CS^n \subset H_{n+i} CS^n \to H_{n+i} QS^n = H_{n+i} \Omega^{i-n} QS^i \xrightarrow{\sigma^{i-n}} H_{2i} QS^i.$$

By properties (A) and (C) this map takes both $Q^i \eta_* s_n$ and $R^i \eta_* s_n$ to $(\eta_* s_i)^2 \neq 0$. The result follows.

Next we must consider a reduced version of $\overline{R}^i_2$. Let $D_2 X$ denote $EZ_2^+ \wedge_{Z_2} (X \wedge X)$. The maps i and d_2 pass to the quotient to give

$$\iota\colon X \wedge X \to D_2 X \quad \text{and} \quad \delta_2 \colon D_2(X \wedge Y) \to D_2 X \wedge D_2 Y.$$

Let $\tilde{R}^i$ be the composite

$$\tilde{H}_n X \subset H_n X \xrightarrow{\overline{R}^i_2} H_{n+i} E_2 X \to \tilde{H}_{n+i} D_2 X,$$

where the last arrow is induced by the evident quotient map $E_2X \to D_2X$. By Lemmas 1 and 2 we have

(1) $$\tilde{R}^n x = \iota_*(x \wedge x) \quad \text{if } x \in \tilde{H}_n X$$

and

(2) $$(\delta_2)_* \tilde{R}^i(x \wedge y) = \sum_{j \geq 0} \tilde{R}^j x \wedge \tilde{R}^{i-j} y.$$

The space $D_2(X^+)$ is homeomorphic to $(E_2X)^+$, and it is easy to see that the composite

$$H_n X = \tilde{H}_n X^+ \xrightarrow{\tilde{R}^i} \tilde{H}_{n+i} D_2(X^+) = H_{n+i} E_2 X$$

is $\overline{R}^i_2$. Thus to prove the theorem it suffices to show $\tilde{R}^i = \tilde{Q}^i$ for all i.

LEMMA 6. *$\tilde{R}^i s_n = \tilde{Q}^i s_n$ for all nonnegative i and n.*

PROOF. The quotient map $E_2X \to D_2X$ factors as

$$E_2 X \to F_2 CX \to F_2 CX / F_1 CX \simeq D_2 X.$$

Since $H_* F_2 CX \to H_* CX$ is monic, we see from Lemma 5 that $\overline{R}^i_2 s_n$ and $\overline{Q}^i_2 s_n$ have equal images in $H_{n+i} F_2 CS^n$. Hence they have equal imges in $\tilde{H}_{n+i} D_2 S^n$.

Next recall from [6] that any element $x \in \tilde{H}_n X$ is represented by a based map

$$h_x \colon S^{n+m} \to K(Z_2, m) \wedge X$$

for some m. By Lemma 6 we have

(3) $$(\delta_2)_* \tilde{R}^{i+m} (h_x)_* s_{n+m} = (\delta_2)_* \tilde{Q}^{i+m} (h_x)_* s_{n+m}$$

in $\tilde{H}_* K(Z_2, m) \otimes \tilde{H}_* X$. We wish to compute the components of each side of (3) in $\tilde{H}_{2m} K(Z_2, m) \otimes \tilde{H}_{n+i} X$. Let ε_m^* be the fundamental class of $\tilde{H}^m K(Z_2, m)$ and let ε_m be its dual. The definitions imply that the slant product $\varepsilon_m^* \backslash (h_x)_* s_{n+m}$ is x, hence $(h_x)_* s_{n+m} = \varepsilon_m \otimes x$ modulo higher degrees in $\tilde{H}_* K(Z_2, m)$. Now equation (2) gives

$$(\delta_2)_* \tilde{R}^{i+m} (h_x)_* s_{n+m} = \tilde{R}^m \varepsilon_m \otimes \tilde{R}^i x$$

modulo higher degrees in $\tilde{H}_* D_2 K(Z_2, m)$. The same calculation holds for $\tilde{Q}^{i+m}$, so we have

(4) $$\tilde{R}^m \varepsilon_m \otimes \tilde{R}^i x = \tilde{Q}^m \varepsilon_m \otimes \tilde{Q}^i x.$$

But by equation (1) we have $\tilde{R}^m \varepsilon_m = \tilde{Q}^m \varepsilon_m = \iota^*(\varepsilon_m \wedge \varepsilon_m)$, and the latter element is nonzero by [**1**, Theorem I.4.1]. Thus equation (4) implies the theorem when $p = 2$.

When p is odd the argument must be modified somewhat. The number 2 should be replaced by p throughout and all two-fold cartesian and smash products $X \times X$ and $X \wedge X$ replaced by p-fold products. The operation R^i increases degrees by $2i(p-1)$ instead of by i. In addition, the following specific changes are needed. Lemma 1 should read:

If $x \in H_{2n} X$ then

$$\overline{R}^n_j = \begin{cases} i_* x^p & \text{if } j = p, \\ 0 & \text{otherwise.} \end{cases}$$

In the proof of Lemma 5 we may assume that $2i > n$, and the map σ^{i-n} should be replaced by

$$\sigma^{2i-n}\colon H_{n+2i(p-1)}\Omega^{2i-n}QS^{2i} \to H_{2pi}QS^{2i}.$$

Equation (1) should be replaced by

$$\tilde{R}^n x = i_*(x^{(p)}) \quad \text{if } x \in \tilde{H}_{2n}X.$$

In the argument following Lemma 6 the map h_x should go from S^{n+2m} to $K(Z_p, 2m) \wedge X$ for some m. The elements s_{n+m} and ε_m in that argument should be replaced by s_{n+2m} and ε_{2m}. To conclude the argument one computes the components of each side of equation (3) in the group $\tilde{H}_{2pm}K(Z_p, 2m) \otimes \tilde{H}_{n+2i(p-1)}X$.

References

1. F. R. Cohen, T. J. Lada and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Math., vol. 533, Springer-Verlag, Berlin and New York, 1976.

2. E. Dyer and R. K. Lashof, *Homology of iterated loop spaces*, Amer. J. Math. **84** (1962), 35–88.

3. J. P. May, *A general algebraic approach to Steenrod operations*, The Steenrod Algebra and its Applications (F. P. Peterson, ed.), Lecture Notes in Math., vol. 168, Springer-Verlag, Berlin and New York, 1970, pp. 153–231.

4. ______, *The geometry of iterated loop spaces*, Lecture Notes in Math., vol. 271, Springer-Verlag, Berlin and New York, 1972.]

5. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Studies, no. 50, Princeton Univ. Press, Princeton, N. J., 1962.

6. G. W. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227–283.

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