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COHOMOLOGY MORPHISMS FROM $H^*(BU;\ Z/p)$ TO $H^*(BZ/p;\ Z/p)$

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ABSTRACT. In this paper we use Hopf algebra and generating function methods to determine the group of all cohomology morphisms from $H^*(BU;\ Z/p)$ to $H^*(BZ/p;\ Z/p)$ that preserve the Steenrod operations, where p is an odd prime. The group $[BZ/p, BU]$ of homotopy classes of maps from BZ/p to BU, which can be calculated directly, is seen to be naturally isomorphic to the group of cohomology morphisms. For $BZ/2$ and BO with coefficients in $Z/2$ there are precisely similar results.

Introduction. Let BZ/p be the classifying spaces of Z/p where p is an odd prime. Recent advances have shown that the mod p cohomology of BZ/p is very special as an algebra over the Steenrod algebra. In this regard, the set of cohomology morphisms from $H^*(X;\ Z/p)$ to $H^*(BZ/p;\ Z/p)$ that preserve the Steenrod operations is of particular interest for a space X.

In this paper we use elementary methods to determine the group of such cohomology morphisms when $X = BU$, the representing space for reduced K-theory on connected spaces. Since a graded algebra morphism

$$H^*(BU;\ Z/p) \to H^*(BZ/p;\ Z/p)$$

is determined by a power series in $1 + xZ/p[[x]]$, this algebraic problem can be handled efficiently by using generating function techniques similar to those developed in my thesis **[1]** (and independently by Bullett and MacDonald **[2]**). In terms of power series, the result is as follows.

THEOREM. *The collection of all algebra morphisms that preserve the Steenrod operations is isomorphic to the complete subgroup of* $1 + xZ/p[[x]]$ *generated by* $1 + kx$ *for* $0 < k < p$.

The complete subgroup above has an interesting description in terms of the p-adic integers $Z_p^\wedge$. The abelian group $1 + xZ/p[[x]]$ is a module over Z via exponentiation. In fact it is a module over $Z_p^\wedge$ via exponentiation; as an example, for $a = \sum a[i]p^i$ in $Z_p^\wedge$, $(1 + kx)^a = \prod(1 + kx^{p^i})^{a[i]}$. The complete subgroup described in the theorem is actually the free module over $Z_p^\wedge$ with basis $1 + kx$ for $0 < k < p$.

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Each homotopy class of maps from BZ/p to BU induces a cohomology morphism. But the group $[BZ/p, BU]$ of homotopy classes of maps from BZ/p to BU can be computed via K-theory, giving the following result.

COROLLARY. *The group $[BZ/p, BU]$ of based maps up to homotopy from BZ/p to BU is isomorphic to the group of all cohomology morphisms that preserve the Steenrod operations.*

PROOF OF THE COROLLARY. For the classifying space of a finite group, (representable) K-theory is isomorphic to the inverse limit K-theory [**7**], so that Atiyah's calculation of the inverse limit K-theory of the classifying space of a finite group applies. Thus $[BZ/p, BU]$ is isomorphic to the augmentation ideal in the completed representation ring of Z/p. But this is isomorphic to the ideal $(w - 1)$ in the cyclotomic ring $Z_p^\wedge[w]/(1 + w + \cdots + w^{(p-1)})$, where $Z_p^\wedge$ denotes the p-adic integers; see [**8**], for example. Thus $[BZ/p, BU]$ is isomorphic to the free $Z_p^\wedge$-module with basis $w^k - 1$ for $0 < k < p$. Here the $w^k - 1$ are represented by the maps $BZ/p \to BU$ coming from the group homomorphisms from Z/p to the group of pth roots of unity in $U(1)$. But $w^k - 1$ in $[BZ/p, BU]$ induces the map in cohomology determined by $1 + kx$ in $1 + xZ/p[[x]]$, since this is the total Chern class of the corresponding line bundle. Thus the proof of the corollary is DONE.

The theorem also yields a simple result for morphisms from $H^*(BU(n))$. Consider the projection $H^*(BU) \to H^*(BU(n))$ induced by the map $BU(n) \to BU$. In the representation of graded algebra morphisms $H^*(BU) \to H^*(BZ/p)$ as power series, an algebra morphism $H^*(BU(n)) \to H^*(BZ/p)$ corresponds to a polynomial of degree $\leqslant n$ in $1 + xZ/p[[x]]$. But the only polynomials of degree $\leqslant n$ in the complete group generated by $1 + kx$ for $0 < k < p$ are those of the form $\prod(1 + kx)^{n[k]}$ where the $n[k]$ are nonnegative integers whose sum is less than or equal to n.

COROLLARY. *The collection of all algebra morphisms $H^*(BU(n)) \to H^*(BZ/p)$ that preserve the Steenrod operations is isomorphic to the collection of all polynomials of the form $\prod(1 + kx)^{n[k]}$ with $\sum n[k] \leqslant n$.*

Note that each such morphism comes from a sum of line bundles on BZ/p.

For $p = 2$ there are precisely similar results, concerning BO rather than BU. The proof of the theorem for odd primes translates to one for $p = 2$, with considerable simplification, giving

THEOREM. *The collection of all those graded algebra morphisms from $H^*(BO; Z/2)$ to $H^*(BZ/2; Z/2)$ that preserve the Steenrod operations is isomorphic to the complete subgroup of $1 + xZ/2[[x]]$ generated by $1 + x$. It follows that the morphisms $H^*(BO(n); Z/2) \to H^*(BZ/2; Z/2)$ that preserve the Steenrod operations correspond to the powers $(1 + x)^m$ with $0 \leqslant m \leqslant n$.*

As above, $[BZ/2, BO]$ can be computed directly, as the inverse limit of $\widetilde{KO}(RP^n)$ for instance; $[BZ/2, BO]$ is isomorphic to $Z_2^\wedge$ with generator $w - 1$, where w

represents the canonical line bundle. Thus $w - 1$ has total Stiefel-Whitney class $1 + x$ and we have

COROLLARY. *The group* $[BZ/2, BO]$ *is naturally isomorphic to the group of all cohomology morphisms that preserve the Steenrod operations.*

The proof of the main theorem is especially of interest through its use of unusual generating function methods in characterizing the action of the Steenrod operations. I understand that K. Ishiguro also has a direct proof of these results using entirely different arguments. The results also follow as a special case of recent work of J. Lannes [**5**], which builds on fundamental discoveries of H. Miller [**6**] and G. Carlsson [**3**] concerning the special properties of the cohomology of BZ/p over the Steenrod algebra. Perhaps generating function methods will be useful in understanding these special properties.

I would like to thank J. Neisendorfer for suggesting the problem, J. Huard for helping with the number theory in §3, and J. Ucci for comments on K-theory.

1. In this section we use generating functions and Hopf algebra structure to describe the action of the Steenrod operations on $H^*(BU; Z/p)$. Recall that $H^*(BU;Z/p)$ is the polynomial algebra $Z/p[c[1], c[2], \dots]$ where $c[n]$ is the universal Chern class in degree $2n$. Let $c(s)$ denote the total Chern class

$$c(s) = \sum c[n]s^n \quad (\text{with the convention that } c[0] = 1),$$

where s is an indeterminate. The comultiplication in $H^*(BU; Z/p)$ is determined by $\Psi c(s) = c(s) \otimes c(s)$; in other words, the formal power series $c(s)$ is "grouplike."

Let $P(t)$ denote the total reduced power operation

$$P(t) = \sum P^n t^n,$$

where t is an indeterminate. $P(t)$ acts naturally and multiplicatively on the Z/p-cohomology of spaces, and the action of the Steenrod algebra on $H^*(BU; Z/p)$ is completely determined by the formal power series $P(t)(c(s))$, which is also grouplike since

$$\Psi P(t)(c(s)) = P(t) \otimes P(t)(\Psi c(s)) = P(t)(c(s)) \otimes P(t)(c(s)).$$

The following elementary lemma gives a general method for discovering identities between grouplike formal power series and similar kinds of comultiplicative expressions in a coalgebra. The basic idea is that two grouplike expressions must be equal if they have the same image under h, where h is a "test morphism" as described in Lemma 1.1.

Consider the problem of classifying coalgebra morphisms from a connected graded coalgebra C to a coalgebra D. For instance, a grouplike formal power series in s and t, such as $P(t)(c(s))$, can be considered as a coalgebra morphism from the graded dual of $Z/p[s, t]$ to $H^*(BU; Z/p)$.

LEMMA 1.1. *Suppose C and D are coalgebras with C connected, and let h: $D \to L$ be linear and injective on the primitives of D. Then two coalgebra morphisms f and g from C to D are equal if and only if $h \circ f = h \circ g$.*

PROOF. Let $h \circ f = h \circ g$. Suppose f is not equal to g; let c be a lowest degree nonzero element on which f and g differ. Then it follows that $f(c) - g(c)$ is primitive in D. But $h(f(c) - g(c)) = 0$, so $f(c) - g(c) = 0$, which is a contradiction and Lemma 1.1 is DONE.

The coalgebra $H^*(BU; Z/p)$ has the following convenient test morphism. Recall that $H^*(BU(1); Z/p)$ is the polynomial algebra $Z/p[x]$ where x has degree 2; the standard map of $BU(1)$ into BU induces an algebra morphism

$$h\colon H^*(BU; Z/p) \to H^*(BU(1); Z/p)$$

such that $h(c(s)) = 1 + sx$. That h is injective on the set of primitives in $H^*(BU; Z/p)$ follows from the splitting principle, or from the self-dual structure of $H^*(BU; Z/p)$ as a Hopf algebra.

Using the above ideas, a simple calculation leads to the following description of the action of the Steenrod operations on $H^*(BU; Z/p)$.

THEOREM 1.1. *$P(t)(c(s)) = \prod c(u_i)$, where the indeterminates $u_1, \ldots, u_p$ satisfy the identity*

$$(*) \qquad \prod (1 + u_i x) = 1 + sx + stx^p.$$

PROOF. First consider the image of the grouplike formal power series $P(t)(c(s))$ under the test morphism h. Since h preserves the Steenrod operations,

$$h(P(t)(c(s))) = P(t)(h(c(s))) = P(t)(1 + sx) = 1 + s(x + tx^p).$$

Next consider $\prod c(u_i)$. It is symmetric in $u_1, \ldots, u_p$ and so can be expressed in terms of elementary symmetric expressions of them; but by the defining identity of $u_1, \ldots, u_p$, the elementary symmetric expressions in them are monomials in s and t. Thus $\prod c(u_i)$ gives a formal power series in s and t. It is clearly grouplike in $H^*(BU; Z/p)$. Since h is multiplicative, $h(\prod c(u_i)) = \prod(1 + u_i x)$. Requiring that the u_i satisfy the defining identity forces h to agree on the two grouplike series. Thus, by Lemma 1.1, Theorem 1.1 is DONE.

2. Let p be an odd prime. The inclusion of Z/p into $U(1)$ induces an embedding of $H^*(BU(1); Z/p)$ as the even-dimensional subalgebra of $H^*(BZ/p; Z/p)$. Thus for an even-dimensional algebra like $H^*(BU; Z/p)$, studying algebra morphisms to $H^*(BZ/p; Z/p)$ is the same as studying algebra morphisms to $H^*(BU(1); Z/p)$. In the remainder of this paper we study the collection of those algebra morphisms from $H^*(BU; Z/p)$ to $H^*(BU(1); Z/p)$ that preserve the Steenrod operations, and prove in this form the first theorem stated in the introduction. The arguments are easily modified (and simplified) to prove the corresponding theorem for BO and $Z/2$.

Consider a power series $f(x) = \sum f[m]x^m$ in $1 + xZ/p[[x]]$; $f(x)$ determines a graded algebra morphism F from $H^*(BU; Z/p)$ to $H^*(BU(1); Z/p)$ by sending $c[m]$ to $f[m]x^m$. The correspondence between $f(x)$ and F gives a bijection between power series and algebra morphisms since $H^*(BU; Z/p)$ is freely generated by the $c[m]$.

DEFINITION. Let S' denote the set of those power series $f(x)$ such that the corresponding algebra morphism F preserves the Steenrod operations. Since $P(t)$ and F are multiplicative and the $c[n]$ generate $H^*(BU; Z/p)$ as an algebra, we have that $f(x)$ is in S' if and only if $F \circ P(t)(c(s)) = P(t) \circ F(c(s))$.

First we ascertain the structure of S'.

LEMMA 2.1. (a) *S' is a subgroup of the multiplicative group* $1 + xZ/p[[x]]$.
(b) *S' is complete with respect to the* (x)*-adic topology on* $Z/p[[x]]$.

PROOF OF (a). If F and G are algebra morphisms then their convolution $F*G$ is defined by

$$(**) \qquad H^*(BU) \xrightarrow{\Psi} H^*(BU) \otimes H^*(BU)$$

$$\xrightarrow{F \otimes G} H^*(BU(1)) \otimes H^*(BU(1)) \to H^*(BU(1)),$$

so that $F * G(c[n]) = \sum F(c[i])G(c[j])$. Thus convolution corresponds to multiplication of the corresponding power series. We see from $(**)$ that if F and G preserve the Steenrod operations, then so also does $F * G$, since the comultiplication in $H^*(BU)$ and the multiplication in $H^*(BU(1))$ are induced by maps of spaces. Thus S' is closed under the multiplication of power series.

If F is an algebra morphism, then $F \circ X$ is its inverse with respect to convolution, where X: $H^*(BU) \to H^*(BU)$ is the Hopf algebra conjugation. If F preserves the Steenrod operations then so does $F \circ X$, since X is induced by a map of spaces. Thus S' is closed with respect to multiplicative inverses.

PROOF OF (b). Suppose $f_n(x)$ converges to $f(x)$ with each $f_n(x)$ in S'. Then the corresponding algebra morphisms F_n converge to F in the sense that, when restricted to elements less than a given degree in $H^*(BU; Z/p)$, F agrees with F_n for n sufficiently large. But for any $c[m]$, $P(t)(c[m])$ has degree less than pm by excess. So for n sufficiently large,

$$F(P(t)(c[m])) = F_n(P(t)(c[m])) = P(t)(F_n(c[m])) = P(t)(F(c[m])).$$

Thus $F \circ P(t)(c(s)) = P(t) \circ F(c(s))$ and Lemma 2.1 is DONE.

The description of the Steenrod operations on $H^*(BU; Z/p)$ from §1 leads to the following simple characterization of S'.

LEMMA 2.2. *$f(x)$ is in S' if and only if*

$$(***) \qquad \prod f(u_i x) = f(sx + stx^p)$$

where $u_1, \ldots, u_p$ satisfy $\prod(1 + u_i x) = 1 + sx + stx^p$.

PROOF. Since $P(t)(x) = x + tx^p$ in $H^*(BU(1); Z/p)$ we see that

$$P(t) \circ F(c(s)) = \sum f[n]s^n(x + tx^p)^n = f(sx + stx^p).$$

But by Theorem 1.1, $P(t)(c(s)) = \prod c(u_i)$, so that $F \circ P(t)(c(s)) = \prod f(u_i x)$. Thus $F \circ P(t)(c(s)) = P(t) \circ F(c(s))$ if and only if $(***)$, and Lemma 2.2 is DONE.

As an example, let $0 < k < p$. Substituting kx for x in $(*)$ shows immediately that $f(x) = 1 + kx$ satisfies $(***)$, so it is in S'.

DEFINITION. Let S denote the complete multiplicative group generated by $1 + kx$ for $0 < k < p$. Note that S is isomorphic as abelian group to the product of $p - 1$ copies of the additive group of p-adic integers.

By the above example and Lemma 2.1 we have that S is included in S'. In the remainder of the paper we show that in fact $S = S'$.

3. In order to prove that S is all of S' we show in Theorem 3.1 that S' is not too big, and in Theorem 3.2 that S is not too small.

THEOREM 3.1. *If $f(x)$ and $g(x)$ are in S' and $f[kp^n] = g[kp^n]$ for every $n \geqslant 0$ and every $0 < k < p$ then $f(x) = g(x)$.*

PROOF. Suppose $f(x)$ is not equal to $g(x)$ and consider the smallest N where $f[N]$ is not equal to $g[N]$. Expanding (***) for $f(x)$ and taking the coefficient of x^N on both sides gives

$$\sum f[I]u^I = \sum \binom{m}{n} f[m]s^m t^n$$

summed over $I = (i_1, \ldots, i_p)$ with $i_1 + \cdots + i_p = N$, and m and n with $m + pn = N$. Here $f[I] = f[i_1] \cdots f[i_p]$ and $u^I = u_1^{i_1} \cdots u_p^{i_p}$. We have a similar result for $g(x)$.

Subtract these expressions for $f(x)$ and $g(x)$. Since $f[i] = g[i]$ for $i < N$, the only terms that do not cancel are those on the left where I has only one nonzero entry, and the one on the right with $m = N$ and $n = 0$. Thus we have

$$(f[N] - g[N]) \sum u_i^N = (f[N] - g[N])s^N.$$

But $f[N] - g[N]$ is nonzero, so $\sum u_i^N = s^N$. The following lemma shows that this can occur only for special values of N.

LEMMA 3.1. *If $\sum u_i^N = s^N$ then $N = kp^n$ for some $0 < k < p$ and some $n \geqslant 0$.*

PROOF. For simplicity, specialize (*) by putting $s = 1$. Thus we assume that

$$(*') \qquad \prod (1 + u_i x) = 1 + x + tx^p.$$

Let $q[N]$ denote the power-sum expression

$$q[N] = \sum u_i^N.$$

Newton's formula determines the $q[N]$ recursively in terms of the coefficients on the right side of (*'), that is, as polynomials in t. In our case, the recursion says:

$q[N] = q[N - 1] + tq[N - p]$ with initial conditions $q[k] = 1$ for $0 < k \leqslant p$. To prove the lemma we must show that $q[N] = 1$ only for special values of N. The following can be checked to be an explicit solution of the recursion:

$$q[N] = \sum \binom{N - 1 - (p - 1)k}{k} s^k.$$

If N is not of the form $N = kp^n$ for some $0 < k < p$, then factoring out the highest power of p dividing N gives $N = (p + r)p^n$ where $r \geqslant 0$ and p does not divide r. It is easy to see that for $k = p^n$ and $N = (p + r)p^n$ the binomial coefficient $\binom{N-1-(p-1)k}{k}$ is congruent to $r \bmod p$; thus $q[N]$ has a nonzero s^{p^n} coefficient, so that $q[N]$ is not equal to 1 and Lemma 3.1 is DONE.

From Lemma 3.1 we see that if $\sum u_i^N = s^N$ then N is one of the coefficients where $f[N] = g[N]$ by assumption, giving a contradiction. Thus Theorem 3.1 is DONE.

To complete the proof that $S = S'$, we need only show that all values of the kp^n coefficients are realized by elements in S.

THEOREM 3.2. *Any choice of coefficients $f[N]$ in Z/p (for $N = kp^n$ with $0 < k < p$ and $n \geqslant 0$) is realized by some $f(x)$ in S.*

PROOF. Let W denote the quotient group resulting when $1 + xZ/p[[x]]$ is truncated at $x^p = 0$. Let T_n denote the function $1 + Z/p[[x]] \to W$ defined by $f(x)$ goes to $\sum f[kp^n]x^k$ summed over $0 < k < p$. Thus $T = T_0$ is the standard truncation homomorphism. We prove the theorem by showing that

$$\prod T_n\colon S \to \prod W$$

is a surjection.

Given $(h_0(x), h_1(x), h_2(x), \ldots)$ in $\prod W$, we say that $f(x)$ in S is good up to level N if $T_n(f(x)) = h_n(x)$ for $n < N$. Clearly 1 is good up to level 0. The following two lemmas gives the inductive step needed to finish the proof.

LEMMA 3.2. *$T\colon S \to W$ is a surjection.*

PROOF. Consider the elements $1 + kx$ for $0 < k < p$. To show that they generate W as a group, we look at their logarithmic derivatives. As in the theory of Witt vectors, making a vector of the coefficients of $x^1, \ldots, x^{p-1}$ in $L(f) = -xf'/f$ gives an isomorphism from W to $p - 1$ copies of Z/p (see [**4**, §4], for instance). Since the coefficient vectors of $L(1 + kx)$ form the rows of a Vandermonde matrix, they are linearly independent and span; thus the $1 + kx$ for $0 < k < p$ span W and Lemma 3.2 is DONE.

LEMMA 3.3. *If $f(x)$ is good up to level N then $f(x)g(x^{p^N})$ is good up to level $N + 1$, where $g(x)$ in S satisfies $T(g(x)) = h_N(x)/T_N(f(x))$ in W. (Such a $g(x)$ exists by Lemma 3.2.)*

PROOF. For $n < N$, $T_n(f(x)g(x^{p^N})) = T_n(f(x))$. Consider $T_n(f(x)g(x^{p^N}))$. Let $(f(x))[m]$ denote the coefficient of x^m in $f(x)$. Since $(g(x^{p^N}))[m] = 0$ unless m is a multiple of p^N, it follows that

$$\left(f(x)g(x^{p^N})\right)[kp^N] = \sum (f(x))[ip^N]\left(g(x^{p^N})\right)[jp^N],$$

which equals

$$\sum (f(x))[ip^N](g(x))[j] = T_N(f(x))T(g(x)) = h_N(x),$$

and Lemma 3.3 is DONE.

The product of the successive $g(x^{p^N})$ produced above converges in S, yielding an element of S which maps to the sequence $(h_n(x))$ in $\prod W$. Thus Theorem 3.2 is DONE.

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