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$\beta([0,\infty))$ DOES NOT CONTAIN NONDEGENERATE HEREDITARILY INDECOMPOSABLE CONTINUA

MICHEL SMITH

ABSTRACT. Bellamy has shown that if $A = [0,\infty)$, then $\beta A - A$ is an indecomposable continuum and every nondegenerate subcontinuum of $\beta A - A$ can be mapped onto every metric continuum. Thus, it follows that every nondegenerate subcontinuum of $\beta A - A$ contains a nondegenerate indecomposable continuum. We show, however, that no nondegenerate subcontinuum of $\beta A - A$ is hereditarily indecomposable. Thus, every nondegenerate subcontinuum of $\beta A - A$ contains a decomposable continuum as well as a nondegenerate indecomposable continuum.

We show that if $A = [0,\infty)$, then $\beta A - A$ does not contain nondegenerate hereditarily indecomposable continua. This fact is surprising in view of one of Bellamy's results. Bellamy **[B]** showed that every nondegenerate subcontinuum of $\beta A - A$ can be mapped onto every metric continuum. Thus, if I is a nondegenerate subcontinuum of $\beta A - A$, then I can be mapped onto any metric hereditarily indecomposable continuum. Furthermore, since I can be mapped onto a nondegenerate indecomposable continuum, it follows that I must contain a nondegenerate indecomposable continuum.

It has been shown **[S]**, however, that if G is a countable discrete collection of nondegenerate hereditarily indecomposable metric continua and $X = \bigcup G$, then every component of $\beta X - X$ is hereditarily indecomposable, and there exist nondegenerate components of $\beta X - X$. Thus, there exists a wealth of examples of metric spaces whose Stone-Čech remainders contain nondegenerate hereditarily indecomposable continua.

DEFINITIONS AND NOTATION. Suppose that X is a topological space. Then $C(X)$ denotes the space of subcontinua of X. If $O_1, O_2, \ldots, O_n$ is a finite collection of open sets in X, then $\mathrm{Rm}^X(\{O_1, O_2, \ldots, O_n\}) = \{H \in C(X) \mid H \cap O_i \neq \varnothing$ for all $i \in \{1,2,\ldots,n\}$ and $H \subset \bigcup_{i=1}^n O_i\}$. The superscript X will be suppressed if it is clear which space is meant. If g and $\hat{g}$ are two collections of open sets in X, then $g \wedge \hat{g}$ denotes the collection, $\{O \cap (\bigcup \hat{g}) \mid O \in g\} \cup \{(\bigcup g) \cap O \mid O \in \hat{g}\}$. Note that $g \wedge \hat{g} = \hat{g} \wedge g$. Suppose g, h, and k are collections of open sets. Then

$$(g \wedge h) \wedge k = \{O \cap (\bigcup k) \mid O \in g \wedge h\} \cup \{O \cap (\bigcup (g \wedge h)) \mid O \in k\}.$$

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Since

$$g \wedge h = \{O \cap (\bigcup h) \mid O \in g\} \cup \{O \cap (\bigcup g) \mid O \in h\},$$

then

$$\begin{aligned}\bigcup(g \wedge h) &= (\bigcup\{O \cap (\bigcup h) \mid O \in g\}) \cup (\bigcup\{O \cap (\bigcup g) \mid O \in h\}) \\ &= ((\bigcup g) \cap (\bigcup h)) \cup ((\bigcup h) \cap (\bigcup g)) = (\bigcup g) \cap (\bigcup h).\end{aligned}$$

Therefore,

$$\begin{aligned}(g \wedge h) \wedge k &= \{O \cap (\bigcup k) \mid O \in g \wedge h\} \cup \{O \cap ((\bigcup g) \cap (\bigcup h)) \mid O \in k\} \\ &= \{O \cap (\bigcup k) \mid O \in \{\hat{O} \cap (\bigcup g) \mid \hat{O} \in h\}\} \\ &\quad \cup \{O \cap (\bigcup k) \mid O \in \{\hat{O} \cap (\bigcup h) \mid \hat{O} \in g\}\} \\ &\quad \cup \{O \cap ((\bigcup g) \cap (\bigcup h)) \mid O \in k\} \\ &= \{\hat{O} \cap (\bigcup g) \cap (\bigcup k) \mid \hat{O} \in h\} \cup \{\hat{O} \cap (\bigcup h) \cap (\bigcup k) \mid \hat{O} \in g\} \\ &\quad \cup \{O \cap (\bigcup g) \cap (\bigcup h) \mid O \in k\}.\end{aligned}$$

Therefore

$$\begin{aligned}(g \wedge h) \wedge k = \{O \cap (\bigcup g) \cap (\bigcup k) \mid O \in h\} &\cup \{O \cap (\bigcup h) \cap (\bigcup k) \mid O \in g\} \\ &\cup \{O \cap (\bigcup g) \cap (\bigcup h) \mid O \in k\}.\end{aligned}$$

So it can be easily verified that $(g \wedge h) \wedge k = g \wedge (h \wedge k)$. So $\wedge$ is associative and the expression $g \wedge h \wedge k$ is well defined. Notice also that $g \wedge h$ refines both g and h and if g and h both cover the continuum H, then $g \wedge h$ also covers H.

If H is a subset of X and g is a collection of open sets that covers H, then g is said to properly cover H if and only if each set in g intersects H. The closure of H in X is denoted by $\mathrm{Cl}_X(H)$.

If X is a metric space, then the Stone-Čech compactification βX can be associated with the set of ultrafilters of closed sets of X [**W**]. Thus if $x \in \beta X$, then x is an ultrafilter of closed sets of X. If $x \in X$, then x is identified with the ultrafilter that contains $\{x\}$.

Suppose that X is a space, $Y = \beta X - X$, and A is a collection of subsets of X. Then $\mathrm{Ls}^X(A)$ is the set to which p belongs if and only if $p \in Y$ and every open set in βX containing p intersects every element of A. Again, the superscript X will be suppressed if it is clear which space is meant. The following theorem follows easily from the definitions.

THEOREM 1. *Suppose that X is a space, $Y = \beta X - X$, and A is a collection of subsets of X. Then $\mathrm{Ls}^X(A)$ is a closed subset of Y.*

The following theorem about the hyperspace of $\beta X - X$ will be needed.

THEOREM 2. *Suppose that X is a locally compact, locally connected, connected metric space, $Y = \beta X - X$, $K \in C(Y)$, and $\{O_1, O_2, \ldots, O_n\}$ is a finite collection of open sets in βX properly covering K. Then there is an element $H \in C(X)$ so that $H \in \mathrm{Rm}^X(\{O_1, O_2, \ldots, O_n\})$.*

PROOF. Recall the following facts about X (see [E, M]):

(i) X is separable,

(ii) if G is a collection of open sets in X and $U = \bigcup G$, then some countable subcollection of G covers U, and

(iii) every component of an open set in X is open.

Suppose that the theorem is not true, and that there exist X and $K \in C(Y)$ satisfying the hypothesis of the theorem and a finite collection of open sets $\{O_1, O_2, \ldots, O_n\}$ in βX that properly covers K, so that $C(X) \cap \mathrm{Rm}^X(\{O_i\}_{i=1}^n) = \varnothing$. Let $x \in K \cap O_1$ and let L be a closed subset of X with $L \in x$ and $L \subset O_1$. By conditions (i) and (ii), local compactness and local connectedness, $O_1 \cap X$ is the union of countably many open sets $\{D_i\}_{i=1}^{\infty}$, so that for each positive integer i, $\mathrm{Cl}_X(D_i)$ is a compact subset of $O_1 \cap X$, and D_i is connected.

Suppose that for some integer i and each $j \in \{2, 3, \ldots, n\}$, there is a subcontinuum H_j of $X \cap (\bigcup_{l=1}^n O_l)$ that intersects D_i and O_j. Thus, $H = (\bigcup_{j=2}^n H_j) \cup \mathrm{Cl}_X(D_i)$ is a subcontinuum of $X \cap (\bigcup_{i=1}^n O_i)$ and $H \cap O_j \neq \varnothing$ for all $j \in \{1, 2, \ldots, n\}$, but this contradicts our original assumption that the theorem is false. Therefore, for each positive integer i, there exists an integer $J \in \{2, 3, \ldots, n\}$ so that every subcontinuum of X that intersects D_i and lies in $\bigcup_{l=1}^n O_l$ misses O_J. (Note that if $D_i \cap L \neq \varnothing$, then $\mathrm{Cl}_X(D_i)$ is a subcontinuum of $X \cap O_1$ containing $L \cap D_i$, and if H is a subcontinuum of $X \cap (\bigcup_{i=1}^n O_i)$ intersecting D_i, then $\mathrm{Cl}_X(D_i) \cup H$ is a subcontinuum of $X \cap (\bigcup_{l=1}^n O_l)$ which contains $D_i \cap L$.) For each positive integer i, let J_i be the set to which j belongs, if and only if $j \in \{2, 3, \ldots, n\}$ and every subcontinuum of $X \cap (\bigcup_{l=1}^n O_l)$ that intersects D_i does not intersect O_j. Suppose J is a subset of $\{2, 3, \ldots, n\}$. Then let $M_J = \bigcup\{D_i \cap L \mid J_i = J\}$.

CLAIM 2.1. *For each* $J \subset \{2, 3, \ldots, n\}$, M_J *is closed.*

PROOF. Suppose p is a limit point of M_J. Then $p \in L$, so $p \in O_1$ and $p \in D_e$ for some positive integer e. So D_e contains a point q of M_J, and $q \in D_k$ for some k so that $D_k \cap L \subset M_J$ and $J_k = J$. But $\mathrm{Cl}_X D_k \cup \mathrm{Cl}_X D_e$ is a subcontinuum of $X \cap (\bigcup_{j=1}^n O_j)$. So, if $j \in \{2, 3, \ldots, n\}$, there is a subcontinuum of $X \cap (\bigcup_{l=1}^n O_l)$ intersecting D_k and O_j if and only if there is one intersecting D_e and O_j. Thus, $J_e = J_k = J$. So, $D_e \cap L \subset M_J$ and $p \in M_J$. This establishes the claim.

Since each point of L lies in O_1, and hence lies in some D_i, we have $L = \bigcup_J M_J$ with the union taken over all $J \subset \{2, 3, \ldots, n\}$. Since there are only finitely many such J, there exists $\hat{J} \subset \{2, 3, \ldots, n\}$ so that $M_{\hat{J}} \in x$. Let $\hat{O}_1 = \bigcup\{D_i \mid J_i = \hat{J}\}$, so $M_{\hat{J}} \subset \hat{O}_1$. Let $\hat{j} \in \hat{J}$, $y \in K \cap O_{\hat{j}}$, and let $\hat{L}$ be a closed subset of X in y, so that $\hat{L} \subset O_{\hat{j}}$. Let $U = \bigcup\{D \mid D$ is a component of $X \cap (\bigcup_{l=1}^n O_l)$ that intersects $\hat{O}_1\}$, and $V = (\bigcup_{i=1}^n O_i - U) \cap X$. Note that $\hat{O}_1 \subset U$, and U is open.

CLAIM 2.2. $\hat{L} \cap U = \varnothing$.

PROOF. Suppose that $\hat{L}$ intersects U. Then $\hat{L}$ intersects a component C of $(\bigcup_{i=1}^n O_i) \cap X$, and C contains D_e for some e, so that $J_e = \hat{J}$. But then there is an arc α lying in C that intersects $\hat{L}$ and D_e. But then α intersects $O_{\hat{j}}$ and D_e, so $\hat{j} \notin J_e$, which contradicts the fact that $\hat{j} \in \hat{J} = J_e$. So Claim 2.2 is established.

It follows from Claim 2.2 that $\hat{L} \subset V$.

CLAIM 2.3. *V is open in X.*

PROOF. Suppose that V is not open and p is a point of V, which is a limit point of $X - V$. But $U \cup V$ is open in X, so p is a limit point of U. Since $U \cup V = (\bigcup_{j=1}^{n} O_j) \cap X$, then some component C of $(\bigcup_{j=1}^{n} O_j) \cap X$ contains p, and since p is limit point of U, we have $C \cap U \neq \varnothing$. So C is a component of $(\bigcup_{i=1}^{n} O_i) \cap X$ that intersects $\hat{O}_1$, so $C \subset U$. So $p \in U$, and hence $p \notin V$ since $V = (\bigcup_{j=1}^{n} O_j - U) \cap X$. This is a contradiction, so the claim is established.

Let $\tilde{U} = \{z \in \beta X \,|\, \text{some set in } z \text{ lies in } U\}$ and $\tilde{V} = \{z \in \beta X \,|\, \text{some set in } z \text{ lies in } V\}$. Then $\tilde{U}$ and $\tilde{V}$ are open in βX.

CLAIM 2.4. $K \subset \tilde{U} \cup \tilde{V}$.

PROOF. Let $z \in K$. Then z contains a closed set L_z of X such that $L_z \subset O_i \cap X$ for some $i \in \{1, 2, \ldots, n\}$, since $\{O_i\}_{i=1}^{n}$ covers K. So $L_z \subset (\bigcup_{i=1}^{n} O_i) \cap X$. But $L_z \cap U$ and $L_z \cap V$ are both closed in X, since U and V are disjoint open sets, so either $L_z \cap U \in z$, or $L_z \cap V \in z$. So either $z \in \tilde{U}$ or $z \in \tilde{V}$, which establishes the claim.

But by Claim 2.2, $y \in \tilde{V}$, and since $M_{\hat{\jmath}} \in x$ and $M_{\hat{\jmath}} \subset \hat{O}_1$, we also have $x \in \tilde{U}$. But $K \subset \tilde{U} \cup \tilde{V}$, and $\tilde{U}$ and $\tilde{V}$ are disjoint. This contradicts the connectedness of K. So the theorem must be true.

EXAMPLE. Theorem 2 does not hold if the condition of local connectivity is relaxed. This is not difficult to see if we let W be the $\sin(1/x)$ continuum in the plane:

$$W = \{(x, y) \,|\, y = \sin(1/x), 0 < x \leqslant 1\} \cup \{(x, y) \,|\, x = 0, -1 \leqslant y \leqslant 1\},$$

and we let $X = W - \{(0, 0)\}$. Then the theorem does not hold with $K = W^* = \beta X - X$ and the covering $\{O_i\}_{i=1}^{4}$ where $\{O_i\}_{i=1}^{4}$ is defined in terms of the open sets $\{U_i\}_{i=1}^{4}$ in X as follows:

$$O_i = \{z \in \beta X \,|\, z \text{ contains a closed set that lies in } U_i\}$$

and let

$$A = \{(x, y) \in X \,|\, x < \tfrac{1}{4} \text{ and } x < y < \tfrac{1}{2}\},$$

$$B = \{(x, y) \in X \,|\, x < \tfrac{1}{4} \text{ and } -2x < y < 2x\},$$

$$C = \{(x, y) \in X \,|\, x < \tfrac{1}{4} \text{ and } -\tfrac{1}{2} < y < -x\},$$

$$E = \{(x, y) \in X \,|\, \pi/2 + 2n\pi < 1/x < 3\pi/2 + 2n\pi \text{ for some positive integer } n \text{ or } x = 0\},$$

$$F = \{(x, y) \in X \,|\, 3\pi/2 + 2n\pi < 1/x < 5\pi/2 + 2n\pi \text{ for some positive integer } n \text{ or } x = 0\}.$$

Then let $U_1 = A$, $U_2 = C$, $U_3 = B \cap E$, $U_4 = B \cap F$.

THEOREM 3. *Suppose $X = [0, \infty)$. Then $\beta X - X$ does not contain a nondegenerate hereditarily indecomposable continuum.*

PROOF. Suppose $X = [0, \infty)$, $Y = \beta X - X$, and $K \in C(Y)$ is nondegenerate. Let A_1, A_2, and A_3 be three distinct points of K. For $i \in \{1, 2, 3\}$ let $\hat{U}_i$ be open in βX with $A_i \in \hat{U}_i$ and $\mathrm{Cl}_{\beta X}\hat{U}_j \cap \mathrm{Cl}_{\beta X}\hat{U}_i = \varnothing$ for $i \neq j$. Furthermore if $i \in \{1, 2, 3\}$, let

U_i and W_i be open in βX, so that

$$A_i \in U_i \subset \mathrm{Cl}_{\beta X} U_i \subset W_i \subset \mathrm{Cl}_{\beta X} W_i \subset \hat{U}_i$$

and every component of $\hat{U}_i \cap X$ contains a component of $U_i \cap X$. ($\hat{U}_i$ can be chosen so that this is possible.)

Let G be the set to which g belongs, if and only if g is a finite collection of open sets covering K, and for each $i \in \{1, 2, 3\}$ some element of g lies in U_i and meets K. If $g \in G$, let $g = \{O_1^g, O_2^g, \ldots, O_{Ng}^g\}$.

CLAIM 3.1. *Suppose $g \in G$. Then there exists an interval $H \in C(X)$ with $H \subset \bigcup g$ and two subintervals α and β with $H = \alpha \cup \beta$, and a permutation σ of $\{1, 2, 3\}$ so that α intersects $U_{\sigma(1)}$ and $U_{\sigma(2)}$ but misses $W_{\sigma(3)}$, β intersects $U_{\sigma(2)}$ and $U_{\sigma(3)}$ but misses $W_{\sigma(1)}$, and $\alpha \cap \beta \subset U_{\sigma(2)}$.*

PROOF. Let $\tilde{g} = \{O \in g \mid O \cap K \neq \varnothing\}$. Then by Theorem 2, there is a continuum $I \in C(X)$ with $I \in \mathrm{Rm}(\tilde{g})$. Let O denote the component of $(\bigcup \tilde{g}) \cap X$ that contains I. Then O is open, connected, and intersects U_i for each $i \in \{1, 2, 3\}$.

Let us assume O is an open interval, and that the end points of O do not lie in $\mathrm{Cl}_X W_i$ for each $i \in \{1, 2, 3\}$; the following proof only requires a slight modification if this is not the case.

If $x \in (\bigcup_{i=1}^3 \mathrm{Cl}_X U_i) \cap O$, then let D_x be an open interval containing x so that $D_x \subset (\bigcup_{i=1}^3 \hat{U}_i) \cap O$. Thus D_x is a subset of one of $\{\hat{U}_i\}_{i=1}^3$. Note that $O \cap (\bigcup_{i=1}^3 \mathrm{Cl}_X U_i)$ is compact. Some finite subcollection Z of $\{D_x \mid x \in O \cap (\bigcup_{i=1}^3 \mathrm{Cl}_X U_i)\}$ covers $O \cap \bigcup_{i=1}^3 \mathrm{Cl}_X U_i$. Let $Z = \{D_i\}_{i=1}^m$, so that for each $i \in \{1, 2, \ldots, m\}$, we have $D_i = D_{x_i}$, and let $\lambda(i)$ denote the integer such that $D_i \subset \hat{U}_{\lambda(i)}$. Since λ is onto $\{1, 2, 3\}$, there must exist a permutation σ of $\{1, 2, 3\}$ and two elements D_k and D_j of Z so that $\lambda(k) = \sigma(1)$, $\lambda(j) = \sigma(3)$, some element D_l of Z lies between D_k and D_j, and if D_e is an element of Z that lies between the open intervals D_k and D_j, then $\lambda(e) = \sigma(2)$. Let $D_k = (a_1, b_1)$, let $D_j = (a_2, b_2)$, and without loss of generality assume $b_1 < a_2$. Thus, $x_k \in (a_1, b_1)$ and $x_j \in (a_2, b_2)$. Let $\alpha = [x_k, x_l]$ and $\beta = [x_l, x_j]$. Since every component of $\hat{U}_i \cap X$ contains a component of $W_i \cap X$ and a component of $U_i \cap X$ for each $i \in \{1, 2, 3\}$, it follows that $H = \alpha \cup \beta$ is the required interval, α and β are the required subintervals, and σ is the required permutation. This establishes Claim 3.1.

Let Ω be the set to which H belongs if and only if $H \subset X$ is an interval and there exists a permutation σ of $\{1, 2, 3\}$ and two subintervals α and β of H with $H = \alpha \cup \beta$ so that

$$\alpha \cap U_{\sigma(1)} \neq \varnothing, \quad \alpha \cap U_{\sigma(2)} \neq \varnothing, \quad \alpha \cap W_{\sigma(3)} = \varnothing,$$
$$\beta \cap U_{\sigma(2)} \neq \varnothing, \quad \beta \cap U_{\sigma(3)} \neq \varnothing, \quad \beta \cap W_{\sigma(1)} = \varnothing,$$

and $\alpha \cap \beta \subset U_{\sigma(2)}$. If $H \in \Omega$ then let $A_{\sigma(1)}^H$ and $A_{\sigma(3)}^H$ denote the intervals α and β respectively and let σ_H denote the permutation in the above definition which corresponds to the interval $H \in \Omega$. If $H \in \Omega$ and $i \in \{1, 2, 3\}$ then let P_i^H be a point of $U_i \cap H$ and require that $P_{\sigma_H(2)}^H \in A_{\sigma_H(1)}^H \cap A_{\sigma_H(3)}^H$. Note that (1) $P_{\sigma_H(1)}^H \notin A_{\alpha_H(3)}^H$ and $P_{\sigma_H(3)}^H \notin A_{\sigma_H(1)}^H$, and (2) $P_{\sigma_H(1)}^H \notin W_{\sigma_H(3)}$ and $P_{\sigma_H(3)}^H \notin W_{\sigma_H(1)}$. If $g \in G$ then let $\Omega g = \{H \in \Omega \mid H \subset \bigcup g\}$. Note that if h refines g then $\Omega h \subset \Omega g$. By Claim 3.1 if $g \in G$ then $\Omega g \neq \varnothing$. Suppose $g \in G$, $i \in \{1, 2, 3\}$, and $\lambda \in \{1, 2, 3\}$. Then let $M_\lambda^i g = \{P_i^H \mid H \in \Omega g$ and $\sigma_H(2) = \lambda\}$. Therefore if $g \in G$, $M_\lambda^i g \subset U_i$ and $M_\lambda^i g \subset \bigcup g$, and for each $P \in M_\lambda^i g$ there exists an interval $H \in \Omega g$ so that $P_i^H = P$ and $\sigma_H(2) = \lambda$.

CLAIM 3.2. *Let $i \in \{1,2,3\}$. Then there exists $\lambda \in \{1,2,3\}$ so that $\mathrm{Cl}_{\beta X}(M_\lambda^i g) \cap K \neq \varnothing$ for all $g \in G$.*

PROOF. Suppose that the claim is not true. Then for each $\lambda \in \{1,2,3\}$, there exists $g_\lambda \in G$ so that $\mathrm{Cl}_{\beta X}(M_\lambda^i g_\lambda) \cap K = \varnothing$. For each $\lambda \in \{1,2,3\}$, let S_λ and T_λ be disjoint open sets in βX so that $\mathrm{Cl}_{\beta X}(M_\lambda^i g_\lambda) \subset S_\lambda$ and $K \subset T_\lambda$. Let $T = \bigcap_{i=1}^3 T_i$ and $h = g_1 \wedge g_2 \wedge g_3 \wedge \{T\}$. Then $h \in G$, and by Claim 3.1 there exists $H \in C(X)$, $H \subset \bigcup h$, and there exists a permutation σ of $\{1,2,3\}$ so that $H \in \Omega h$ and $\sigma = \sigma_H$. Let $k = \sigma(2)$. Then $P_i^H \in M_k^i h \subset M_k^i g_k$. But $H \subset T$, so $P_i^H \in T$, which contradicts the fact that $M_k^i g_k \subset S_k$ and $P_i^H \in M_k^i g_k$.

CLAIM 3.3. *Let $\lambda \in \{1,2,3\}$ be such that $\mathrm{Cl}_{\beta X}(M_\lambda^1 g) \cap K \neq \varnothing$ for all $g \in G$. Then $\mathrm{Cl}_{\beta X}(M_\lambda^j g) \cap K \neq \varnothing$ for all $g \in G$ and all $j \in \{2,3\}$.*

PROOF. Suppose that the claim is not true. Then there exists $g \in G$ and $j \in \{2,3\}$ so that $\mathrm{Cl}_{\beta X}(M_\lambda^j g) \cap K = \varnothing$. Let S and T be disjoint open sets in βX so that $\mathrm{Cl}_{\beta X}(M_\lambda^j g) \subset S$ and $K \subset T$. Let $h = g \wedge \{T\}$. Then $h \in G$, so there exists $H \in C(X)$ such that $H \in \Omega h$ and $\sigma_H(2) = \lambda$. But $H \subset T$ and $j = \sigma(e)$ for some e and $P_j^H \in H \cap U_j$. So $P_j^H \in T$, but $P_j^H \in M_\lambda^j h \subset M_\lambda^j g$, which contradicts the fact that $M_\lambda^j g \subset S$ and $S \cap T = \varnothing$. This establishes the claim.

Suppose λ is a number such that, by Claims 3.2 and 3.3, $\mathrm{Cl}_{\beta X}(M_\lambda^i g) \cap K \neq \varnothing$ for all $g \in G$ and all $i \in \{1,2,3\}$. Let $\theta_\lambda^i = \{M_\lambda^i g \mid g \in G\}$.

CLAIM 3.4. *If $i \in \{1,2,3\}$, then $\mathrm{Ls}(\theta_\lambda^i) \cap K \neq \varnothing$.*

PROOF. Suppose that the claim is not true. Then for each $x \in K$, there is an open set C_x in βX containing x, and an element $g_x \in G$ so that $M_\lambda^i g_x \cap C_x = \varnothing$. Let E_x be an open set in βX so that $x \in E_x \subset \mathrm{Cl}_{\beta X} E_x \subset C_x$, and let V_x be an open set in βX so that $M_\lambda^i g_x \subset V_x$ and $V_x \cap E_x = \varnothing$ (note that $V_x = \beta X - \mathrm{Cl}_{\beta X} E_x$ will work). Some finite subcollection $\{E_{x_i}\}_{i=1}^m$ of $\{E_x \mid x \in K\}$ covers K. Let E be an open set such that

$$K \subset E \subset \mathrm{Cl}_{\beta X} E \subset \bigcup_{i=1}^m E_{x_i}.$$

Let $h = g_{x_1} \wedge g_{x_2} \wedge \cdots \wedge g_{x_m} \wedge \{E\}$. Then by the definition of λ, there exists $H \in C(X)$ so that $H \in \Omega h$ and $\sigma_H(2) = \lambda$.

So $P_i^H \in H$ and $P_i^H \in M_\lambda^i h$. But $H \subset E$, so $P_i^H \in E$ and hence $P_i^H \in E_{x_e}$ for some $e \in \{1,2,\ldots,m\}$. Furthermore, $M_\lambda^i h \subset M_\lambda^i g_{x_e}$ so $P_i^H \in M_\lambda^i g_{x_e} \subset V_{x_e}$, which contradicts $E_{x_e} \cap V_{x_e} = \varnothing$. This establishes Claim 3.4.

The following claim is easy to verify.

CLAIM 3.5. *If $i \in \{1,2,3\}$, then $\mathrm{Ls}(\theta_\lambda^i) \subset \mathrm{Cl}_{\beta X} U_i$.*

Without loss of generality let us assume that $\lambda = 2$. Then if $g \in G$, let

$$Ag = \bigcup\{A_1^H \mid H \in \Omega g \text{ and } \sigma_H(2) = \lambda\}$$

and

$$Bg = \bigcup\{A_3^H \mid H \in \Omega g \text{ and } \sigma_H(2) = \lambda\}.$$

Note that for all $g \in G$:

$$M_\lambda^1 g \subset Ag, \quad M_\lambda^\lambda g = M_\lambda^2 g \subset Ag \cap Bg,$$
$$M_\lambda^3 g \subset Bg, \quad \text{and} \quad Ag \cup Bg \subset \bigcup g.$$

Let $A = \{Ag \mid g \in G\}$ and $B = \{Bg \mid g \in G\}$.

CLAIM 3.6. $\mathrm{Ls}(A) \subset K$ *and* $\mathrm{Ls}(B) \subset K$.

PROOF. Suppose that the claim is not true and $\mathrm{Ls}(A) \not\subset K$. Then there exists a point $y \in \mathrm{Ls}(A)$ so that $y \notin K$. Let V and T be disjoint open sets so that $y \in V$ and $K \subset T$. Let $\hat{g} \in G$ and $\tilde{g} = \hat{g} \wedge \{T\}$. But then V does not intersect $A\tilde{g}$, which contradicts the fact that $y \in \mathrm{Ls}(A)$; therefore, V must intersect Ag for all $g \in G$. Similarly, it follows that $\mathrm{Ls}(B) \subset K$. This establishes Claim 3.6.

Notice from the definitions that

$$\mathrm{Ls}(\theta_\lambda^\lambda) \subset \mathrm{Ls}(A) \quad \text{and} \quad \mathrm{Ls}(\theta_\lambda^\lambda) \subset \mathrm{Ls}(B).$$

Let $u \in \mathrm{Ls}(\theta_\lambda^\lambda)$. Then $u \in K$.

CLAIM 3.7a. *Some subcontinuum of* $\mathrm{Ls}(A)$ *contains u and intersects* $\mathrm{Cl}_{\beta X} U_1$.

PROOF. Suppose that the claim is not true and no subcontinuum of $\mathrm{Ls}(A)$ containing u intersects $\mathrm{Cl}_{\beta X} U_1$. Then $\mathrm{Ls}(A)$ is the union of two disjoint compact sets M and N with $u \in M$ and $\mathrm{Ls}(A) \cap \mathrm{Cl}_{\beta X} U_1 \subset N$. Let V and T be disjoint open sets in βX with $M \subset V$, $N \subset T$, such that $V \cap \mathrm{Cl}_{\beta X} U_1 = \varnothing$.

Let $O_1, O_2, \dots, O_n$ be a finite open covering of $K - (V \cup T)$ such that $\mathrm{Cl}_{\beta X} O_i \cap (N \cup M) = \varnothing$ for all $i \in \{1, 2, \dots, n\}$. Let $\hat{g} = \{T, V, O_1, O_2, \dots, O_n\}$. Now, $\mathrm{Ls}(A) \subset T \cup V$, and from the definition of $\mathrm{Ls}(\theta_\lambda^\lambda)$ and the fact that $g \wedge \hat{g} \in G$ for all $g \in G$, we have $V \cap M_\lambda^\lambda(g \wedge \hat{g}) \neq \varnothing$ for all $g \in G$. But for each $x \in K - (T \cup V)$, $x \notin \mathrm{Ls}(A)$; so there exists an open set O_x and an element $g_x \in G$ such that $x \in O_x$ and $O_x \cap Ag_x = \varnothing$. Some finite subcollection $\{O_{x_i}\}_{i=1}^{e}$ of $\{O_x \mid x \in K - (T \cup V)\}$ covers $K - (T \cup V)$. Let

$$\tilde{g} = \hat{g} \wedge \{T, V, O_{x_1}, O_{x_2}, \dots, O_{x_e}\} \wedge g_{x_1} \wedge \cdots \wedge g_{x_e}.$$

By the definition of u, $V \cap A\tilde{g} \neq \varnothing$, so there exists $H \subset \bigcup \tilde{g}$, $H \in C(X)$, so that $H \in \Omega\tilde{g}$, $\sigma_H(2) = 2$, and $P_\lambda^H \in V$. But $P_1^H \notin V$, since $P_1^H \in U_1$, so $P_1^H \in T$ or $P_1^H \in O_{x_i}$ for some $i \in \{1, 2, \dots, e\}$. Suppose $P_1^H \in O_{x_i}$, $i \in \{1, 2, \dots, e\}$. Then O_{x_i} intersects A_1^H and $A_1^H \subset Ag_{x_i}$, which is a contradiction of the definitions of g_{x_i} and O_{x_i}. Thus, $P_1^H \in T$. Since $H = A_1^H \cup A_3^H$ and A_1^H is a continuum that intersects each of the disjoint open sets T and V, it follows that $A_1^H \cap O_{x_j} \neq \varnothing$ for some $j \in \{1, 2, \dots, e\}$. But $A_1^H \subset Ag_{x_j}$, so O_{x_j} intersects Ag_{x_j}, which is a contradiction. So Claim 3.7a is established.

Similarly we have

CLAIM 3.7b. *Some subcontinuum of* $\mathrm{Ls}(B)$ *contains u and intersects* $\mathrm{Cl}_{\beta X} U_3$.

Let I be a subcontinuum of $\mathrm{Ls}(A)$ that contains u and intersects $\mathrm{Cl}_{\beta X} U_1$, and let J be a subcontinuum of $\mathrm{Ls}(B)$ that contains u and intersects $\mathrm{Cl}_{\beta X} U_3$. We now wish to establish that $I \cup J$ is decomposable.

CLAIM 3.8a. $\mathrm{Ls}(A) \cap \mathrm{Cl}_{\beta X} U_3 = \varnothing$.

PROOF. If $g \in G$, and $H \in \Omega g$ is such that $\sigma_H(2) = 2$, then $H = A_1^H \cup A_3^H$ and $A_1^H \cap W_3 = \varnothing$. Therefore, $\mathrm{Ls}(A) \cap W_3 = \varnothing$, and since $\mathrm{Cl}_{\beta X} U_3 \subset W_3$, we have $\mathrm{Ls}(A) \cap \mathrm{Cl}_{\beta X} U_3 = \varnothing$.

Similarly we have

CLAIM 3.8b. $\mathrm{Ls}(B) \cap \mathrm{Cl}_{\beta X} U_1 = \varnothing$.

Therefore, since $I \subset \mathrm{Ls}(A)$ and $J \subset \mathrm{Ls}(B)$, we have $I \cap \mathrm{Cl}_{\beta X} U_3 = \varnothing$ and $J \cap \mathrm{Cl}_{\beta X} U_1 = \varnothing$. But $I \cap \mathrm{Cl}_{\beta X} U_1 \neq \varnothing$, $J \cap \mathrm{Cl}_{\beta X} U_3 \neq \varnothing$, and $u \in I \cap J$. So $I \cup J$ is decomposable. Therefore, since $I \subset \mathrm{Ls}(A) \subset K$ and $J \subset \mathrm{Ls}(B) \subset K$, then $I \cup J \subset K$. This proves the theorem.

QUESTION. Does Theorem 3 hold in general, that is: if X is a locally compact separable metric space so that $\beta X - X$ is a continuum, and X does not contain a hereditarily indecomposable continuum, then does every nondegenerate subcontinuum of $\beta X - X$ contain a decomposable continuum?

BIBLIOGRAPHY

[**B**] D. P. Bellamy, *A non-metric indecomposable continuum*, Duke Math. J. **38** (1971), 15–20.

[**D**] R. F. Dickman, Jr., *A necessary and sufficient condition for $\beta X - X$ to be an indecomposable continuum*, Proc. Amer. Math. Soc. **33** (1972), 191–194.

[**E**] R. Engelking, *Outline of general topology*, North-Holland, Amsterdam, 1968.

[**M**] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloq. Publ., Vol. 12, Providence, R. I., 1962.

[**S**] M. Smith, *$\beta(X - \{x\})$ for X not locally connected*, Topology Appl. **26** (1987).

[**W**] R. C. Walker, *The Stone-Čech compactification*, Springer-Verlag, Berlin and New York, 1974.

DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36849