k-TO-1 FUNCTIONS ON ARCS FOR k EVEN

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ABSTRACT. For exactly k-to-1 functions from [0,1] into [0,1]:
(1) at least one discontinuity is required (Harrold),
(2) if k = 2, then infinitely many discontinuities are needed, for any Hausdorff
image space (Heath),
(3) if k = 4, or if k is odd, then there is such a function with only one
discontinuity (Katsuura and Kellum),
and, it is shown here that
(4) if k is even and k > 4, then there is such a function with only two
discontinuities, and no such function exists with fewer discontinuities.

I. Introduction. A function is k-to-1 if each point inverse has exactly k elements,
and it is at most k-to-1 if each point inverse has at most k elements.

Over 45 years ago, O. G. Harrold, Jr. proved in [2] that there is no continuous
k-to-1 map from [0,1] into [0,1]. In a recent paper [4], H. Katsuura and K. Kellum
demonstrate that for each odd positive integer k, and for k = 4, there is a k-to-1
function from [0,1] into [0,1] with exactly one discontinuity. They ask what is the
minimum number of discontinuities for k-to-1 functions from [0,1] into [0,1] with k
even. In [3] the author showed that any 2-to-1 function from [0,1] to a Hausdorff
space requires infinitely many discontinuities. Thus only even numbers greater than
4 need be considered. Theorems 1 and 2 answer the question raised by Katsuura and
Kellum for even integers at least 6:

THEOREM 1. If f: [0,1] —> [0,1] is a k-to-1 function and k is an even integer with
k > 4, then f has at least two discontinuities.

THEOREM 2. If k is an even integer greater than 4 then there is a k-to-1 function
from [0,1] into [0,1] with only two discontinuities.

II. Proof of Theorem 1.

LEMMA 1. Suppose f: (0,1) —> (0,1) is a continuous map at most k-to-1, p is in
(0,1), and d is a positive number. Then there is a number x with x < p and
|x - p| < d such that:
1. either f((x, p)) ⊆ (f(x), f(p)), or f((x, p)) ⊆ (f(p), f(x)), and
2. if \( I \) is any subinterval of \((0,1)\) between \( f(x) \) and \( f(p) \), then there is a subinterval \( J \) of \( I \) such that every horizontal line \( \{ y = c \} \) with \( c \) in \( J \) intersects the graph of \( f \) between \( x \) and \( p \) an odd number of times.

**Proof.** Since \( f^{-1}(f(p)) \) is finite, there is a positive number \( d' < d \) such that no point within \( d' \) of \( p \) maps to \( f(p) \) except \( p \). Choose any number \( x' \) less than \( p \) so that \( |x' - p| < d' \). The set \( f^{-1}(f(x')) \) is finite so there is an \( x \) with \( x' \leq x < p \) and \( f(x) = f(x') \) such that no point of \((x, p)\) maps to \( f(x') \). Part 1 is true for this \( x \). Note that the part 1 property implies that each point in \((0,1)\) is either a crossing point, a local maximum, or a local minimum for the graph of \( f \).

Now suppose part 2 is false and suppose \( f(x) < f(p) \). Then there is an interval \( I \) in \((f(x), f(p))\) such that every subinterval contains a number \( c \) where the line \( \{ y = c \} \) intersects the graph between \( x \) and \( p \) an even number of times. Let \( c_1 \) be such a number in \( I \) and let \( n_1 \) be the even number of times \( \{ y = c_1 \} \) intersects the graph between \( x \) and \( p \). Since the graph goes continuously from \((x, f(x))\) below \( \{ y = c_1 \} \) to \((p, f(p))\) above, an odd number of these intersection points must be crossings, say \( j \) of them. Of the others, \( i \) are local minima and \( m \) are local maxima. Since \( m + i \) is odd, one is larger, say \( m > i \). Part 1 is true for each point in \( f^{-1}(c_1) \) so there is an interval \( I_2 = (t, c_1) \) in \( I \) small enough that if \( c \) is in \( I_2 \) then \( \{ y = c \} \) intersects the graph between \( x \) and \( p \) at least \( j + 2m \) times. From the negation of part 2 there is a number \( c_2 \) in \( I_2 \) such that the cardinality of this intersection is even, say \( n_2 \). Then

\[
n_2 > j + 2m > j + m + i = n_1.
\]

This process can be continued until \( n_a > k \). Since the line \( \{ y = c_a \} \) cannot intersect the graph of an at most \( k \)-to-1 map more than \( k \) times this is a contradiction.

**Note 1.** Obviously the same properties hold to the right of \( p \).

**Note 2.** Part 1 is a strengthening of a lemma found in Katsuura and Kellum [4].

Now, to prove Theorem 1, suppose that \( f: [0,1] \to [0,1] \) is a \( k \)-to-1 function, with \( k \) an even integer greater than 4, and \( f \) has only one discontinuity, \( q \).

**Claim 1.** Without loss of generality it can be assumed that \( f([0,1]) \) is connected.

If \( q = 0 \), then, since \( f([0,1]) \) is connected and \( f(q) = f(x) \) for some \( x \) in \((0,1] \), the image of \( f \) is connected. If \( 0 < q < 1 \), the connected sets \( f([0,q]) \) and \( f([q,1]) \) are disjoint, and \( f(q) = f(x) \) for some \( x \) less than \( q \), then \( f \) restricted to \([0,q] \) is also a \( k \)-to-1 function with one discontinuity (Harrold proved there is at least one discontinuity) from \([0, q] \) to \([0,1] \).

**Claim 2.** Without loss of generality it can be assumed that \( f([0,1]) = (0,1) \).

Since the image is connected it is an interval with or without endpoints. Suppose 0 is in the image of \( f \). One of 0 or 1 is not \( q \), say \( 1 \neq q \). Let \( z = 1 \) if \( f(1) = 0 \) and \( z = 0 \) if \( f(1) \neq 0 \). There are \( k \) points that map to 0; \( n \) of them are neither 0, 1, nor \( q \), with \( n > 2 \) since \( k > 5 \). Then \( |f^{-1}(0)| = k \leq n + z + 2 \), where the 2 allows the possibility that \( q \) and 0 are different and both map to 0. Each of the \( n \) points are local minima for \( f \). Since \( f \) restricted to each of \((0,q)\) and \((q,1)\) is continuous and at most \( k \)-to-1, Lemma 1 holds. From part 1 of the lemma used at points of \( f^{-1}(0) \)
there is a number $e$ close to 0 so that the line $\{y = e\}$ intersects the graph of $f$ at least $2n + z$ times. Since it intersects the graph exactly $k$ times the following is true:

$$n + z + 2 \geq k \geq 2n + z,$$

from which it follows that $2 > n$, a contradiction. Therefore 0 is not in the image of $f$.

**Claim 3.** The discontinuity $q$ is in $(0,1)$ and the limit of the images of any sequence converging to $q$ from the left is 0 and from the right is 1 (or the other way around). It was proved in Katsuura and Kellum [4] that each limit exists and is either 0 or 1. If both limits were 1, say, then there would be an interval of numbers $(0, e)$ not mapped onto, contradicting the fact that the image of $f$ is $(0, 1)$. For the same reason, since both 0 and 1 limits must be achieved, $q$ must be interior to $[0, 1]$ to have two sides in the domain.

**Claim 4.** $f(0) = f(1)$. If not, one of them, say $f(1)$, is not equal to $f(q)$. From Lemma 1 there are disjoint intervals $(a_i, b_i), i = 1, 2, \ldots, k - 1$, about the points of $f^{-1}(f(1))$ other than 1 and an interval $(a_k, 1]$ disjoint from the others, that satisfy part 1.

Since points near $q$ map to values near 0 or 1, there is a positive number $e$ so that the $e$-neighborhood about the line $\{y = f(1)\}$ contains no point of the graph of $f$ whose first coordinate lies outside the intervals $(a_i, b_i)$ and $(a_k, 1)$.

From part 2 of the lemma there are numbers $c$ in $(f(1), f(1) + e)$ and $c'$ in $(f(1) - e, f(1))$ so that if $x_i$ is the point of $f^{-1}(f(1))$ in $(a_i, b_i)$ then

1. if $x_i$ is a crossing point for the graph of $f$ on the line $\{y = f(1)\}$, then both of the lines $\{y = c\}$ and $\{y = c'\}$ intersect the graph of $f$ an odd number of times between $a_i$ and $b_i$,

2. if $x_i$ is a local minimum (or maximum) for $f$ then the line $\{y = c\}$ (or $\{y = c'\}$) intersects the graph of $f$ an even number of times between $a_i$ and $b_i$, and the other line $\{y = c'\}$ (or $\{y = c\}$) does not intersect the graph between $a_i$ and $b_i$, and

3. one of the lines $\{y = c\}$ and $\{y = c'\}$ intersects the graph an odd number of times between $a_k$ and 1 and the other not at all. It will be assumed here that $\{y = c\}$ is the one that intersects the graph over $(a_k, 1]$.

Let $m, n,$ and $j$ denote the number of local maxima, local minima, and crossing points in $f'^{-1}(f(1)) - \{1\}$, respectively. Then

$$|f^{-1}(c)| = k$$

is the sum of $m$ even numbers plus $j + 1$ odd numbers, and

$$|f'^{-1}(c)| = k$$

is the sum of $m$ even numbers and $j$ odd numbers. Since $k$ cannot be both even and odd there is a contradiction.

**Claim 5.** If Claim 3 is as stated and not the other way around, then $f((0, q)) \subseteq (0, f(0))$, and $f((q, 1)) \subseteq [f(1), 1)$.

Suppose there is a point $(a, b)$ on the graph of $f$ with $0 < a < q$, $f(0) < b$, and $b$ is the largest such value. The inverse of $b$ contains $m$, say, local maxima points, and the number of crossing points, $j$, must be odd since they are all greater than $q$ and the graph of $f$ from $q$ to 1 goes from $y$-coordinates arbitrarily close to 1 near $q$ down to $f(1)$ at 1, and $f(1) < b$. As in Claim 4, there is a number $c' < b$, $c' \neq f(q)$,
close enough to \( b \) that \( |f^{-1}(c')| = k \) is the sum of \( m \) even numbers and \( j \) odd numbers, which is impossible for the even number \( k \).

**Claim 6.** There is a contradiction.

With the graph of \( f \) to the left of \( q \) below and on \( \{ y = f(0) \} \) and the graph to the right of \( q \) above and on the same line, all of the points (\( m \) of them) of \( f^{-1}(f(0)) \) in \((0, q)\) are local maxima. Again, choose \( c' \) near \( f(0) \) with \( c' < f(0), c' \neq f(q) \), and the line \( \{ y = c' \} \) intersects the graph in \( k \) points, the sum of an odd number, corresponding to the point \((0, f(0))\), plus \( m \) even numbers; a contradiction.

**III. Proof of Theorem 2.** The construction here is a generalization of an example in [4] which in turn used examples from [1] and [2]. The basic pieces of the graph of the function to be constructed are generalized \( W \)'s and \( M \)'s defined for a given integer \( n \):

(i) Choose \( 2n + 1 \) distinct points in \([0,1]\), \( 0 = a_0 < a_1 < \cdots < a_{2n} = b \), and define \( W(a_{2t}) = 1 \) and \( W(a_{2t+1}) = 0 \) for relevant \( t \). Denote by \( W(n) \) the piecewise linear extension to \([0,1]\).

(ii) Given an interval \([a, b]\), choose \( 2n + 2 \) points \( a = a_0 < a_1 < \cdots < a_{2n+1} = b \), and define \( M(a_{2t}) = a \) and \( M(a_{2t+1}) = b \) for relevant \( t \). Let \( M(a, b, n) \) denote the piecewise linear extension to a function from \([a, b]\) onto \([a, b]\). These basic pieces will be used to describe the basic units \( B(n) \) and \( A(t, n) \) for integers \( t < n \):

(iii) Define the function \( B_1(n) \) from \([0,1]\) to \([0,1]\) by

\[
 B_1(n) = (0,0) + \sum_{i=0}^{\infty} M(2^{-i+1}, 2^{-i}, n).
\]

Get the graph of \( B_2(n) \) by first reflecting the graph of \( B_1(n) \) about the vertical line \( \{ x = .5 \} \) and then about the horizontal line \( \{ y = .5 \} \).

(iv) Divide the rectangle \([0,3] \times [0,2]\) into six unit squares using matrix \( S(i, j) \) notation. In the lower left square, \( S(2,1) \), put the graph of \( B_3(m) \); in \( S(2,2) \) put the graph of \( W(t) \); and in the upper right square, \( S(1,3) \), put the graph of \( B_2(n) \).

Consider the cardinality, \( I(c) \), of the intersection of the line \( \{ y = c \} \) with this composite graph: If \( 1 < c < 2 \) then \( I(c) = 2n + 1 \). \( I(1) = m + t + n + 1 \). If \( 0 < c < 1 \) then \( I(c) = 2m + 1 + 2t \).

For the interior intersections to be constant, \( 2n + 1 = m + t + n + 1 = 2m + 1 + 2t \) is needed, i.e. \( m + t = n \).

Given \( n \) and \( t \), then, set \( m = n - t \) and denote the described function on \([0,3]\) by \( A_1(t, n) \). Denote by \( A_2(t, n) \) the function constructed by reflecting the graph of \( A_1(t, n) \) about the line \( \{ x = 1.5 \} \), and by \( A_3(t, n) \) the function constructed by reflecting the graph of \( A_1(t, n) \) about \( \{ y = 1 \} \).

Finally, since \( k \) is an even integer greater than 4 it can be written as \( (2r + 1) + (2s + 1) \) where \( r \) and \( s \) are both positive. Define a continuous function \( g \) from \([0, 12]\) to \([0,4]\) by dividing \([0, 12] \times [0,4]\) into eight \( 2 \times 3 \) rectangles. Put \( A_4(r - 1, r) \) in the upper left rectangle \( T(1,1) \), \( A_2(s - 1, s) \) in \( T(1,2) \), \( A_3(r - 1, r) \) in \( T(2,3) \), and \( B_1(s) \) in the lower right rectangle \( T(2,4) \).
The wanted function $f$ will equal $g$ everywhere except at its two discontinuities, $f(3) = 2$ and $f(9) = 2$. The careful reader will agree that every point inverse has $k$ elements.

REFERENCES

2. O. G. Harrold, Jr., Exactly $(k,1)$ transformations on connected linear graphs, Amer. J. Math. 62 (1940), 823–834.

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