SHORTER NOTES

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FOURIER SERIES WITH POSITIVE COEFFICIENTS

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ABSTRACT. Extending a result of N. Wiener, it is shown that functions on the circle with positive Fourier coefficients that are pth power integrable near 0, 1 < p ≤ 2, have Fourier coefficients in lp.

The following result was proved (but never published) by Norbert Wiener in the early 1950's. (See [1, pp. 242, 250] and [3].)

WIENER'S THEOREM. If \( \sum c_ne^{int} \) is the Fourier series of a function \( f \in L^1(-\pi, \pi) \) with \( c_n \geq 0 \) for all \( n \), and \( f \) restricted to a neighborhood \((-\delta, \delta)\) of the origin belong to \( L^2(-\delta, \delta) \), then \( f \) belongs to \( L^2(-\pi, \pi) \).

A question which immediately arises in connection with this result is the following: does the theorem remain true if one replaces \( L^2(-\delta, \delta) \) and \( L^2(-\pi, \pi) \) in its statement respectively by \( L^p(-\delta, \delta) \) and \( L^p(-\pi, \pi) \), with \( 1 < p \leq \infty \)? In 1969 Stephen Wainger showed, by ingenious counterexamples, that the answer is negative for \( 1 < p < 2 \) [4]. If \( p \) is an even integer or \( \infty \) it is very easy to see that the answer is "yes." For every other exponent between 2 and \( \infty \) it is "no," as was shown in 1975 by Harold S. Shapiro [3]. These negative results have been extended to compact abelian groups [2]. However, the conclusion of Wiener's theorem can be stated equivalently as "then \( \sum c_n^2 < \infty \)." This suggested the following theorem.

THEOREM. If \( \sum c_ne^{int} \) is the Fourier series of a function \( f \in L^1(-\pi, \pi) \) with \( c_n \geq 0 \) for all \( n \), and \( f \) restricted to a neighborhood \((-\delta, \delta)\) of the origin belongs to \( L^p(-\delta, \delta) \) with \( 1 < p < 2 \), then \( \sum c_n^{p'} < \infty \), where \( p' = p/(p - 1) \).

PROOF. (See [3, p. 12].) Let \( h(t) \) be the \( 2\pi \)-periodic function which for \( |t| \leq \pi \) is defined by

\[
    h(t) = \begin{cases} 
    1 - |t|/\delta, & \text{if } |t| \leq \delta, \\
    0, & \delta \leq |t| \leq \pi.
    \end{cases}
\]

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Then \(|hf| \leq \chi_{[-\delta,\delta]}|f| \in L^p(-\pi, \pi)\) and \(\sum |(hf)^\wedge(n)|^p'\) is finite by the Hausdorff-Young inequality. (See (*) below.) We have
\[
(h \cdot f)^\wedge(n) = \sum_{k+l=n} \hat{h}(k)c_l
\]
where \(c_l \geq 0\) for all \(l\) by hypothesis and \(\hat{h}(k) \geq 0\) for all \(k\) by direct calculation. Drop all terms except the \(k = 0\) one from the right side of the last equation to get
\[
c_n \leq \frac{(h \cdot f)^\wedge(n)}{\hat{h}(0)} = \frac{2\pi}{\delta} (h \cdot f)^\wedge(n).
\]
Take \(p'\)th powers and sum over \(n\).

REMARKS. 1. This theorem was motivated by studying the above-mentioned counterexamples of Wainger [4].

2. It is very well known that the Hausdorff-Young theorem consists of two irreversible implication, one of which is

(*) if \(\sum c_ne^{inx}\) is the Fourier series of a function \(f \in L^p(-\pi, \pi)\), where \(1 < p < 2\), then \(\{c_n\} \in l^{p'}\). (See [5, pp. 101–103].)

Wainger's counterexamples are functions designed to satisfy the hypotheses of our theorem while violating the hypothesis of (*). Our theorem shows that they must also satisfy the conclusion of (*), and thereby gives another demonstration that the converse of (*) is false.

3. Our theorem easily extends to compact abelian groups. In the above proof simply replace \([-\pi, \pi]\) by a general compact abelian group and \([-\delta, \delta]\) by a symmetric neighborhood of the identity, note that \(h = \varphi \ast \tilde{\varphi}\) where \(\varphi(t) = \tilde{\varphi}(-t) = 1/\sqrt{\delta} \chi_{[-\delta/2,\delta/2]}(t)\), etc.

REFERENCES