PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 3, November 1987

# THE OCCURRENCE PROBLEM FOR MAPPING CLASS GROUPS

JOHN STILLWELL

(Communicated by Thomas J. Jech)

ABSTRACT. The occurrence problem for the mapping class group $M(g,0)$ of the closed orientable surface of genus $g$ is shown to be solvable for $g = 1$ and unsolvable for $g > 1$.

The occurrence problem for a finitely presented group $G$ is the problem of deciding, given $w, u_1, \ldots, u_n \in G$ (written as words in the generators of $G$), whether $w \in \langle u_1, \ldots, u_n \rangle$, the subgroup of $G$ generated by $u_1, \ldots, u_n$. Since the occurrence problem has the word problem as a special case (to decide whether $w = 1$, ask whether $w \in \langle 1 \rangle$), it is also known as the generalized word problem. The latter term is due to Magnus, who solved the problem for one-relator groups in [**6**].

Mikhailova [**9**] showed that the occurrence problem is unsolvable for $F_2 \times F_2$, the direct product of two copies of the free group $F_2$ of rank 2. Using this result, Makanina [**8**] showed that the occurrence problem is also unsolvable for the $n$-string braid groups $B_n$ when $n \geq 5$. Since braid groups are related to mapping class groups, one might expect a similar result for mapping class groups. The present paper confirms this, showing that the occurrence problem is unsolvable for $M(g,0)$, the mapping class group of the closed orientable surface of genus $g$, when $g \geq 2$. As far as I know, this is the first appearance of unsolvability in surface topology (the conjugacy problem for mapping class groups was solved by Hemion [**2**]).

To show that the value of $g$ is best possible, we also show

THEOREM 1. *The occurrence problem is solvable for $M(1,0)$, the mapping class group of the torus.*

What makes $M(1,0)$ special is the fact that it has a certain quotient group for which the solution of the occurrence problem is already known, from [**1** or **10**]. Following a suggestion of the referee, I shall first show how solvability of the occurrence problem for a group $G$ follows from solvability of the occurrence problem for a quotient $G/N$ in fairly general circumstances.

In order to have a precise setting in which to employ algorithms, let us assume that the generators of $G/N$ are written as barred letters $\bar{x}$, where the letters $x$ are the generators in a fixed finite presentation of $G$. This implies, in particular, that we can go effectively from a word $\bar{w} = \bar{x}_1\bar{x}_2 \cdots \bar{x}_n$ for an element of $G/N$ to a preimage $w = x_1 x_2 \cdots x_n$ in $G$, simply by erasing the bars. An instance of the occurrence problem for $G/N$ is a question whether $\bar{w} \in \langle \bar{u}_1, \ldots, \bar{u}_n \rangle$, where $\bar{w}, \bar{u}_1, \ldots, \bar{u}_n$ are

Received by the editors August 13, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20F10; Secondary 57M05.
*Key words and phrases.* Occurrence problem, mapping class group.

©1987 American Mathematical Society
0002-9939/87 $1.00 + $.25 per page

words in the letters $\bar{x}$. A solution of the occurrence problem for $G/N$ is then an algorithm which processes any such words $\bar{w}, \bar{u}_1, \ldots, \bar{u}_n$ and correctly answers whether $\bar{w} \in \langle \bar{u}_1, \ldots, \bar{u}_n \rangle$. With this understanding of the problem and what its solvability means we assume

(1) The occurrence problem for $G/N$ is solvable.

A natural complement to this assumption is

(2) The occurrence problem for $G$ is solvable when restricted to instances $v \in \langle u_1, \ldots, u_n \rangle$ with $v \in N$.

LEMMA. *Under conditions* (1) *and* (2), *the occurrence problem for G is solvable.*

PROOF. Given an instance "$w \in \langle u_1, \ldots, u_n \rangle$?" of the occurrence problem for $G$, we first use the solution of the occurrence problem for $G/N$ to decide whether $\bar{w} \in \langle \bar{u}_1, \ldots, \bar{u}_n \rangle$. If this is not the case, then of course $w \notin \langle u_1, \ldots, u_n \rangle$.

If, on the other hand, $\bar{w} \in \langle \bar{u}_1, \ldots, \bar{u}_n \rangle$, then we can find a specific word $\bar{u}_{i_1} \cdots \bar{u}_{i_k}$ equal to $\bar{w}$ by enumerating all such words, and for each one testing whether $u_{i_1} \cdots \bar{u}_{i_k} \cdot \bar{w}^{-1} \in \langle 1 \rangle$ until we get the answer yes, again using the solution of the occurrence problem for $G/N$. Having found $\bar{u}_{i_1} \cdots \bar{u}_{i_k}$, we compute the preimages $u_{i_1}, \ldots, u_{i_k}, w$ in $G$ of $\bar{u}_{i_1}, \ldots, \bar{u}_{i_k}, \bar{w}$, and consider the element

$$u_{i_1} \cdots u_{i_k} w^{-1} = v \in N.$$

Obviously, $v \in \langle u_1, \ldots, u_n \rangle \Rightarrow w \in \langle u_1, \ldots, u_n \rangle$ and conversely $v \notin \langle u_1, \ldots, u_n \rangle \Rightarrow w = v^{-1} u_{i_1} \cdots u_{i_k} \notin \langle u_1, \ldots, u_n \rangle$. Hence $w \in \langle u_1, \ldots, u_n \rangle \Leftrightarrow v \in \langle u_1, \ldots, u_n \rangle$ and we can decide the latter question by assumption (2). $\square$

PROOF OF THEOREM 1. It is a classical result that $M(1,0) = \mathrm{SL}(2, \mathbf{Z})$. A matrix $\binom{m\ n}{p\ q} \in \mathrm{SL}(2, \mathbf{Z})$ faithfully represents the class of the torus mapping which sends the usual latitude curve $a$ and meridian curve $b$ to $ma + nb$ and $pa + qb$ respectively. We write (homotopy classes of) curves additively here since the fundamental group of the torus is abelian. Conversely, all orientation-preserving torus mappings, whose classes comprise $M(1,0)$ by definition, are represented by matrices in $\mathrm{SL}(2, \mathbf{Z})$ (Tietze [**12**, p. 89]).

It is also known (see e.g. [**7**, p. 47]) that $\mathrm{SL}(2, \mathbf{Z})$ can be generated by the matrices

$$S = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and that the following relations between $S$ and $T$ define $\mathrm{SL}(2, \mathbf{Z})$ abstractly:

$$S^3 T^2 = T^4 = 1.$$

If $W$ is any element of $\mathrm{SL}(2, \mathbf{Z})$ and if $-W$ denotes the negative of $W$ (as a matrix), then the cosets $\overline{W} = \{\pm W\}$ are the elements of the quotient group

$$\mathrm{PSL}(2, \mathbf{Z}) = \mathrm{SL}(2, \mathbf{Z})/\{\pm I\},$$

where $I$ is the identity matrix. Since $S^3 = T^2 = -I$, we get $\overline{S}^3 = \overline{T}^2 = 1$, and these are in fact the defining relations of $\mathrm{PSL}(2, \mathbf{Z})$, first found by Klein and Fricke [**3**, p. 454].

These relations show that $\mathrm{PSL}(2, \mathbf{Z})$ is the free product of $\mathbf{Z}_3 = \langle \overline{S}; \overline{S}^3 = 1 \rangle$ and $\mathbf{Z}_2 = \langle \overline{T}; \overline{T}^2 = 1 \rangle$, and Mikhailova [**10**] and Boydron [**1**] give algorithms for the occurrence problem in such free products. Thus Theorem 1 will follow from the

Lemma if we can show that assumption (2) holds for the subgroup $N = \{\pm I\}$ of $\mathrm{SL}(2, \mathbf{Z})$. This amounts to finding an algorithm which decides, given $u_1, \ldots, u_n \in \mathrm{SL}(2, \mathbf{Z})$ (written as words in $S, T$), whether

$$-I \in \langle u_1, \ldots, u_n \rangle.$$

Finding this algorithm is the hard part of the proof, and it seems to depend heavily on the free product structure of $\mathrm{PSL}(2, \mathbf{Z})$.

In processing $u_1, \ldots, u_n$ we are guided by the behavior of their cosets $\bar{u}_1, \ldots, \bar{u}_n$ in $\mathrm{PSL}(2, \mathbf{Z})$. Since $\mathrm{PSL}(2, \mathbf{Z}) \simeq \mathbf{Z}_3 * \mathbf{Z}_2$, the theorem of Kurosh [**4**] says that the subgroup $\langle \bar{u}_1, \ldots, \bar{u}_n \rangle$ is the free product of a free group $F$ and conjugates of the subgroups $\mathbf{Z}_3$ and $\mathbf{Z}_2$ generated by $\overline{S}$ and $\overline{T}$ (since the latter have no proper nontrivial subgroups). Lyndon [**5**] gives an algorithm for converting $\{\bar{u}_1, \ldots, \bar{u}_n\}$ to a basis $\{\overline{u_1^*}, \ldots, \overline{u_m^*}\}$ which reflects the free product structure of $\langle \bar{u}_1, \ldots, \bar{u}_n \rangle$. That is, $\overline{u_m^*}, \ldots, \overline{u_m^*}$ consists of free generators of $F$, together with generators of the other free factors of $\langle \bar{u}_1, \ldots, \bar{u}_n \rangle$. The latter generators are therefore conjugates of $\overline{S}$ or $\overline{T}$.

Lyndon's algorithm is a sequence of Nielsen transformations, hence the same sequence of transformations, applied to $\{u_1, \ldots, u_m\}$, will also give a generating set for $\langle u_1, \ldots, u_n \rangle$. The only point to bear in mind is that transformations which yield 1 as a generator in $\langle \bar{u}_1, \ldots, \bar{u}_n \rangle$ may yield $-I$ in $\langle u_1, \ldots, u_n \rangle$. The 1 will of course be omitted from the basis of $\langle \bar{u}_1, \ldots, \bar{u}_n \rangle$, whereas $-I$ must not be omitted from the generating set for $\langle u_1, \ldots, u_n \rangle$. However, if $-I$ arises in this way (as we can recognize by multiplying out the matrices which $S, T$ denote) we know immediately that $-I \in \langle u_1, \ldots, u_n \rangle$, and there is no need to continue with construction of $\{\overline{u_1^*}, \ldots, \overline{u_m^*}\}$. If it does not arise, then Lyndon's algorithm on $\{u_1, \ldots, u_n\}$ yields a generating set $\{u_1^*, \ldots, u_m^*\}$ of $\langle u_1, \ldots, u_n \rangle$ corresponding to the free product basis $\{\overline{u_1^*}, \ldots, \overline{u_m^*}\}$ of $\langle \bar{u}_1, \ldots, \bar{u}_n \rangle$.

Now if $-I \in \langle u_1, \ldots, u_n \rangle$ we have an equation

(i) $$-I = u_{i_1}^* u_{i_2}^* \cdots u_{i_l}^*$$

which implies

($\bar{\text{i}}$) $$1 = \overline{u_{i_1}^* u_{i_2}^*} \cdots \overline{u_{i_l}^*}.$$

Since the $\overline{u_i^*}$ form of a free product basis, the normal form theorem for elements of free products [**7**, p. 182] implies that the right-hand side of ($\bar{\text{i}}$) contains a subword of the form either

($\bar{\text{a}}$) $\overline{u_i^*}\,\overline{u_i^*}^{-1}$, or

($\bar{\text{b}}$) $(\overline{u_i^*})^3$ where $\overline{u_i^*} = \overline{A}\,\overline{S}\,\overline{A}^{-1}$, or

($\bar{\text{c}}$) $(\overline{u_i^*})^2$ where $\overline{u_i^*} = \overline{B}\,\overline{T}\,\overline{B}^{-1}$.

Consequently, the right-hand side of (i) contains a subword of the form either

(a) $u_i^* u_i^{*-1}$, or

(b) $(u_i^*)^3$ where $u_i^* = ASA^{-1}$, or

(c) $(u_i^*)^2$ where $u_i^* = BTB^{-1}$.

By cancellation of any adjacent inverses (a), we see that (i) is only possible if the generating set $\{u_i^*, \ldots, u_m^*\}$ contains either an $ASA^{-1}$ or a $BTB^{-1}$. Conversely, if such an element is present, then an equation (i) holds, since

$$(ASA^{-1})^3 = (BTB^{-1})^2 = -I.$$

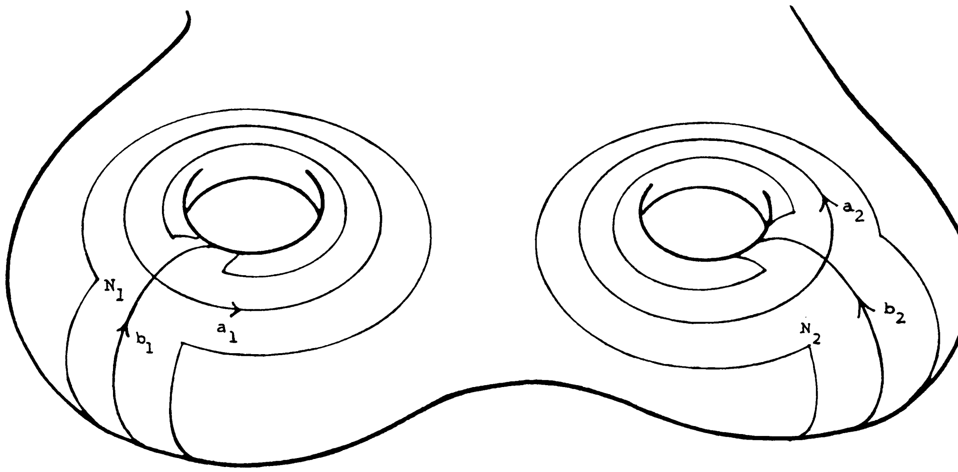


FIGURE 1

To sum up,

$$-I \in \langle u_1, \dots, u_n \rangle \Leftrightarrow \text{ an element } ASA^{-1} \text{ or } BTB^{-1} \in \{u_i^*, \dots, u_m^*\}$$
$$\Leftrightarrow \text{ an element } \overline{A}\,\overline{S}\,\overline{A}^{-1} \text{ or } \overline{B}\,\overline{T}\,\overline{B}^{-1} \in \{\overline{u_i^*}, \dots, \overline{u_m^*}\}.$$

Since we have an algorithm for producing $\{\overline{u_i^*}, \dots, \overline{u_m^*}\}$, and we can decide whether any $\overline{u_i^*}$ is equal to an element $\overline{A}\,\overline{S}\,\overline{A}^{-1}$ or $\overline{B}\,\overline{T}\,\overline{B}^{-1}$ by the algorithm for the conjugacy problem in free products [**7**, p. 188], we have an algorithm for deciding whether $-I \in \langle u_1, \dots, u_n \rangle$, as required. □

The unsolvability result for $g > 1$ which follows uses the topological interpretation of mapping classes, rather than a particular presentation of $M(g, 0)$. Thus it is necessary to mention that finite presentations of $M(g, 0)$ are known. The simplest published so far is in Wajnryb [**13**], and Wajnryb's generators in fact include the twist mappings used below.

THEOREM 2. *The occurrence problem for $M(g, 0)$ is unsolvable when $g > 1$.*

PROOF. Our method of proof is to show that $M(g, 0)$ contains $F_2 \times F_2$ as a subgroup $H$ when $g > 1$. Then there cannot be an algorithm for deciding occurrences of $w$ in the subgroup of $M(g, 0)$ generated by $u_1, \dots, u_n$, otherwise we could apply it to any $w, u_1, \dots, u_n \in H$ to solve the occurrence problem for $F_2 \times F_2$, contrary to Mikhailova [**9**].

We can find two copies of $F_2$ in $M(g, 0)$ as follows. Take two distinct handles of the orientable surface $S_g$ of genus $g > 1$ and consider the curves $a_i, b_i$ in each handle as shown in Figure 1. Let $G_i$ be the subgroup of $M(g, 0)$ generated by twist mappings along $a_i, b_i$. Thus each element of $G_i$ can be represented by a mapping which is the identity outside a union $N_i$ of annular neighborhoods of $a_i, b_i$. By choosing these neighborhoods to be sufficiently thin, we can ensure that $N_1$ is disjoint from $N_2$, as shown in Figure 1. Our representatives of $G_1$ then commute with our representatives of $G_2$. Consequently, each element of $G_1$ commutes with each element of $G_2$, and the subgroup of $M(g, 0)$ generated by $G_1$ and $G_2$ is just $G_1 \times G_2$.

To find an $F_2 \times F_2$ in $M(g,0)$ it therefore suffices to find an $F_2$ in each $G_i$. $G_1$ and $G_2$ are obviously isomorphic so we need look only at $G_1$. Now the twists $t_{a_1}$ and $t_{b_1}$ along $a_1$ and $b_1$ induce automorphisms of the homology group $H_1(S_g)$ of $S_g$ which can be faithfully represented by the matrices $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Namely, $t_{a_1}$ sends $a_1$ to $a_1$, $b_1$ to $a_1+b_1$, and $t_{b_1}$ sends $a_1$ to $a_1+b_1$, $b_1$ to $b_1$, writing (homology classes of) curves additively; both are the identity on the remaining generators $a_i, b_i$ of $H_1(S_g)$. It follows that the subgroup of $G_1$ generated by $t_{a_1}^2$ and $t_{b_1}^2$ has a homomorphic image generated by

$$\begin{pmatrix}1&0\\1&1\end{pmatrix}^2=\begin{pmatrix}1&0\\2&1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1&1\\0&1\end{pmatrix}^2=\begin{pmatrix}1&2\\0&1\end{pmatrix}.$$

However, the latter matrices are known to generate a free group, $F_2$ (this was essentially known to Klein and Fricke [**3**, p. 276], and an algebraic proof may be found in [**7**, p. 101]). Consequently $t_{a_1}^2$ and $t_{b_1}^2$ generate an $F_2$ in $G_1$, as required. □

The referee has pointed out that an $F_2 \times F_2$ in $M(g,0)$, $g \geq 2$, may also be found using known group-theoretic results. This arises from the identification of $M(g,0)$ with the outer automorphism group of

$$\pi_1(S_g)=\langle a_1,b_1,\ldots,a_g,b_g; a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}=1\rangle,$$

established in Nielsen [**11**].

First consider the full automorphism group $\mathrm{Aut}(\pi_1(S_g))$ and, for each $i$, its subgroup $A_i$ which maps the subgroup $\langle a_i, b_i\rangle$ of $\pi_1(S_g)$ onto itself and fixes each $a_j$ and $b_j$ for $j \neq i$. Any two such $A_i$, say $A_1$ and $A_2$, obviously generate their own direct product in $\mathrm{Aut}(\pi_1(S_g))$. That is, if $\alpha_1 \in A_1$ and $\alpha_2 \in A_2$ then $\alpha_1\alpha_2 = \alpha_2\alpha_1$. It follows fairly easily that in the quotient $M(g,0) = \mathrm{Aut}(\pi_1(S_g))/\mathrm{Inn}$, where Inn is the group of inner automorphisms, the corresponding groups $G_1 = A_1\,\mathrm{Inn}/\mathrm{Inn}$, $G_2 = A_2\,\mathrm{Inn}/\mathrm{Inn}$ also generate their own product. This pair $G_1, G_2$ is the same as the pair defined topologically in Theorem 2.

It now remains to show, using the algebraic definition, that each $G_i$ contains an $F_2$. One can either use theorems of Nielsen which show $A_i\,\mathrm{Inn}/\mathrm{Inn} \simeq \mathrm{SL}(2,\mathbf{Z})$ or else observe the action of $G_i$ on $H_1(S_g)$, which is the same as that of $A_i$, since inner automorphisms of $\pi_1(S_g)$ act trivially on its abelianization $H_1(S_g)$. It is clear algebraically, even more so than topologically, that the induced automorphisms of $H_1(S_g)$ include those represented by the matrices $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Thus with either approach one obtains an $F_2$ from the matrices $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$.

## REFERENCES


1. Y. Boydron, *Algorithmes dans les produits libres*, C. R. Acad. Sci. Paris **282** (1976), A135–A138.
2. G. Hemion, *On the classification of homeomorphisms of 2-manifolds and the classification of 3-manifolds*, Acta Math. **32** (1979), 123–155.
3. F. Klein and R. Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunktion*, vol. I, Teubner, Stuttgart, 1890.
4. A. G. Kurosh, *Die Untergruppen der freien Produkte von beliebigen Gruppen*, Math. Ann. **109** (1934), 647–660.
5. R. Lyndon, *Length functions in groups*, Math. Scand. **12** (1963), 209–234.
6. W. Magnus, *Das Identitätsproblem für Gruppen mit einer definierenden Relation*, Math. Ann. **106** (1932), 295–307.

7. W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory*, Wiley-Interscience, New York, 1966.
8. T. A. Makanina, *The occurrence problem in the braid group $B_{n+1}$ with $n+1 \geq 5$*, Mat. Zametki **29** (1981), 31–33. (Russian)
9. K. A. Mikhailova, *The occurrence problem for direct products of groups*, Dokl. Akad. Nauk SSSR **119** (1958), 1103–1105. (Russian)
10. ——, *The occurrence problem for free products of groups*, Mat. Sb. **75** (1968), 199–210.
11. J. Nielsen, *Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen* I, Acta Math. **50** (1927), 189–358.
12. H Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten*, Monatsh. Math. Phys. **19** (1908), 1–118.
13. B. Wajnryb, *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. **45** (1983), 157–174.


DEPARTMENT OF MATHEMATICS, MONASH UNIVERSITY, CLAYTON, VICTORIA 3168, AUSTRALIA