A SIMPLE PROOF OF JACOBI'S FOUR-SQUARE THEOREM

M. D. HIRSCHHORN

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Abstract. Jacobi's four-square theorem, which gives the number of representations of a positive integer as a sum of four squares, is shown to follow simply from the triple-product identity.

1. Recently [2], I showed how one can obtain Jacobi's two-square theorem from the triple-product identity. In this note I show that the triple-product identity also gives Jacobi's four-square theorem:

Theorem 1. The number \( r_4(n) \) of representations of the positive integer \( n \) as a sum of four squares is given by

\[
r_4(n) = 8 \sum_{d \mid n, 4 \nmid d} d.
\]

2. The triple product identity is

\[
\prod_{n \geq 1} \left( 1 + ax^{2n-1} \right) \left( 1 + a^{-1}x^{2n-1} \right) \left( 1 - x^{2n} \right) = \sum_{-\infty}^{\infty} a^n x^n^2.
\]

It follows easily (loc. cit.) that

\[
(1) \quad (a - a^{-1}) \prod_{n \geq 1} \left( 1 - a^{2n} \right) \left( 1 - a^{-2}x^{n} \right) \left( 1 - x^{n} \right) = \sum_{-\infty}^{\infty} (-1)^n a^{2n+1} x^{(n^2+n)/2}.
\]

Differentiate (1) with respect to \( a \), put \( a = 1 \), divide by 2 and we obtain the identity

\[
(2) \quad \prod_{n \geq 1} (1 - x^n)^3 = \frac{1}{2} \sum_{-\infty}^{\infty} (-1)^n (2n + 1) x^{(n^2+n)/2}.
\]

This is a celebrated identity of Jacobi [1, Theorem 357], and can be considered our starting point.

Squaring (2) gives

\[
\prod_{n \geq 1} (1 - x^n)^6 = \frac{1}{4} \sum_{m, n = -\infty}^{\infty} (-1)^{m+n} (2m+1)(2n+1) x^{(m^2+n^2+m+n)/2}.
\]
Now split the sum on the right into two, according as \(m + n\) is even or odd, to obtain
\[
\prod_{n \geq 1} (1 - x^n)^6 = \frac{1}{4} \left\{ \sum_{m=n \pmod{2}} (2m+1)(2n+1)x^{(m^2+n^2+m+n)/2} - \sum_{m \neq n \pmod{2}} (2m+1)(2n+1)x^{(m^2+n^2+m+n)/2} \right\}.
\]
In the first sum, set \(r = \frac{1}{2}(m + n), s = \frac{1}{2}(m - n)\), and in the second set \(r = \frac{1}{2}(m - n - 1), s = \frac{1}{2}(m + n + 1)\), and, remarkably, the two sums coalesce to give
\[
\prod_{n \geq 1} (1 - x^n)^6 = \frac{1}{2} \sum_{r,s=-\infty}^\infty \left( (2r+1)^2 - (2s)^2 \right)x^{r^2+s^2+r}.
\]
Once again splitting the sum, we obtain
\[
\prod_{n \geq 1} (1 - x^n)^6 = \frac{1}{2} \left\{ \sum_{s=-\infty}^\infty x^{s^2} \sum_{r=-\infty}^\infty (2r+1)^2x^{r^2+r} - \sum_{r=-\infty}^\infty x^{r^2+r} \sum_{s=-\infty}^\infty (2s)^2x^{s^2} \right\}
\]
\[
= \frac{1}{2} \left\{ \sum_{s=-\infty}^\infty x^{s^2} \left( 1 + 4x \frac{d}{dx} \right) \sum_{r=-\infty}^\infty x^{r^2+r} - \sum_{r=-\infty}^\infty x^{r^2+r} \times 4x \frac{d}{dx} \sum_{s=-\infty}^\infty x^{s^2} \right\}.
\]
Making use of the triple-product identity, we obtain
\[
\prod_{n \geq 1} (1 - x^n)^6 = \frac{1}{2} \left\{ \prod_{n \geq 1} (1 + x^{2n-1})^2(1 - x^{2n}) \times \left( 1 + 4x \frac{d}{dx} \right) \prod_{n \geq 1} (1 + x^{2n})^2(1 - x^{2n}) - 2\prod_{n \geq 1} (1 + x^{2n})^2(1 - x^{2n}) \times 4x \frac{d}{dx} \prod_{n \geq 1} (1 + x^{2n-1})^2(1 - x^{2n}) \right\}.
\]
Employing the product rule to evaluate the derivatives, we find
\[
\prod_{n \geq 1} (1 - x^n)^6 = \prod_{n \geq 1} (1 + x^{2n-1})^2(1 - x^{2n})(1 + x^{2n})^2(1 - x^{2n}) \times \left( 1 + 8 \sum_{n \geq 1} \frac{2nx^{2n}}{1 + x^{2n}} - 4 \sum_{n \geq 1} \frac{2nx^{2n}}{1 - x^{2n}} \right)
\]
\[
- \prod_{n \geq 1} (1 + x^{2n})^2(1 - x^{2n})(1 + x^{2n-1})^2(1 - x^{2n}) \times \left( 8 \sum_{n \geq 1} \frac{(2n-1)x^{2n-1}}{1 + x^{2n-1}} - 4 \sum_{n \geq 1} \frac{2nx^{2n}}{1 - x^{2n}} \right),
\]
or,
\[
\prod_{n \geq 1} (1 - x^n)^6 = \prod_{n \geq 1} (1 + x^{2n-1})^2(1 + x^{2n})^2(1 - x^{2n})^2 \times \left( 1 - 8 \sum_{n \geq 1} \left( \frac{(2n-1)x^{2n-1}}{1 + x^{2n-1}} - \frac{2nx^{2n}}{1 + x^{2n}} \right) \right).
\]
Dividing both sides by
\[
\prod_{n \geq 1} \left( 1 + x^n \right)^4 (1 - x^n)^2 = \prod_{n \geq 1} (1 + x^n)^2 (1 - x^{2n})^2
\]
\[
= \prod_{n \geq 1} (1 + x^{2n-1})^2 (1 + x^{2n})^2 (1 - x^{2n})^2,
\]
we obtain
\[
\prod_{n \geq 1} \left( \frac{1 - x^n}{1 + x^n} \right)^4 = 1 - 8 \sum_{n \geq 1} \left( \frac{(2n - 1)x^{2n-1}}{1 + x^{2n-1}} - \frac{2nx^{2n}}{1 + x^{2n}} \right).
\]
Now, it is a simple consequence of the triple-product identity that
\[
\prod_{n \geq 1} \left( \frac{1 - x^n}{1 + x^n} \right) = \sum_{-\infty}^{\infty} (-1)^n x^n,
\]
so we have
\[
\left( \sum_{-\infty}^{\infty} (-1)^n x^n \right)^4 = 1 - 8 \sum_{n \geq 1} \left( \frac{(2n - 1)x^{2n-1}}{1 + x^{2n-1}} - \frac{2nx^{2n}}{1 + x^{2n}} \right).
\]
Putting \(-x\) for \(x\), we obtain
\[
\left( \sum_{-\infty}^{\infty} x^n \right)^4 = 1 + 8 \sum_{n \geq 1} \left( \frac{(2n - 1)x^{2n-1}}{1 - x^{2n-1}} + \frac{2nx^{2n}}{1 + x^{2n}} \right)
\]
\[
= \sum_{n \geq 1} \left( \frac{(2n - 1)x^{2n-1}}{1 - x^{2n-1}} + \frac{2nx^{2n}}{1 - x^{2n}} \right) - 8 \sum_{n \geq 1} \left( \frac{2nx^{2n}}{1 - x^{2n}} - \frac{2nx^{2n}}{1 + x^{2n}} \right)
\]
\[
= 1 + 8 \sum_{n \geq 1} \frac{nx^n}{1 - x^n} - 8 \sum_{n \geq 1} \frac{4nx^{4n}}{1 - x^{4n}},
\]
or,
\[
\left( \sum_{-\infty}^{\infty} x^n \right)^4 = 1 + 8 \sum_{4n}^{\infty} \frac{nx^n}{1 - x^n}
\]
from which Theorem 1 follows directly.

**References**


School of Mathematics, University of New South Wales, P. O. Box 1, Kensington, New South Wales, Australia 2033