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ON L_1-CONTRACTION FOR SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. We prove that for 2×2, strictly hyperbolic, genuinely nonlinear systems of conservation laws, there is no metric D such that

$$\int_{-\infty}^{\infty} D(u(x,t),c)\,dx$$

is a nonincreasing function of time for every weak solution u, $u_0(\pm\infty) = c$.

For 2×2, strictly hyperbolic, genuinely nonlinear (cf. [**1**]) systems of conservation laws it was proved in [**2**] that there is no metric D, compatible with the state space, such that

$$I_D(u,v;t) \equiv \int_{-\infty}^{\infty} D(u(x,t),\, v(x,t))\,dx \tag{1}$$

is a nonincreasing function of time for any two weak solutions u, v whose initial conditions agree off a compact set.

In [**2**] a metric D is compatible with the state space Σ if

C1. D: $\Sigma \times \Sigma \to \mathbf{R}$ is a symmetric function.

C2. $D(u,v) + D(v,w) \geqslant D(u,w)\ \forall\, u,v,w \in \Sigma$.

C2. $C_0^{-1}|u - v| \leqslant D(u,v) \leqslant C_0|u - v|\ \forall\, u,v \in \Sigma$ with a uniform constant C_0.

We generalize and simplify the methods in [**2**], and this enables us to relax condition C3.

We wish to point out that relaxing condition C3 is important since it rules out the use e.g. of entropies or quadratic functions to obtain certain integral decay estimates. It is also interesting to note that the solutions used in the construction of the counterexamples below are the elementary "spikes" used frequently in decay arguments (cf. [**3**]).

Thus let

$$u_t + (f(u))_x = 0 \tag{2}$$

be any 2×2 system, strictly hyperbolic, genuinely nonlinear and without coinciding shock and rarefaction curves on a region $N \subset \mathbf{R}^2$. Let $\lambda_1(u)$ and $\lambda_2(u)$ be the eigenvalues of $df(u)$ with corresponding eigenvectors $r_1(u)$ and $r_2(u)$. Let $R_1(u, u^*)$

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and $R_2(u, u^*)$ be respectively the integral curves of $r_1(u)$ and $r_2(u)$ passing through u^*. $R_1(u, u^*)$ and $R_2(u, u^*)$ are called rarefaction curves. Let $S_1(u, u^*)$ and $S_2(u, u^*)$ be the curves of states that can be joined by, respectively, a 1-shock and 2-shock to the right of u^*. These are called shock curves. Given a state u^* on N, shock and rarefaction curves exist locally [**1**].

We then have the following theorem.

THEOREM 1. *Let u and v be weak solutions of* (2) *whose initial conditions agree off a compact set. Then there exists no metric D, which is symmetric, such that $I_D(u, v; t)$ is a nonincreasing function of time.*

PROOF. Take any states u_L, u_R, $\bar{u}$, and $\bar{\bar{u}}$ related in the following way (see Figure 1).

(i) u_R and u_L are joined by a 1-shock with speed s_1, with u_L on the left.
(ii) u_R and $\bar{u}$ are joined by a 1-rarefaction.
(iii) $\bar{u}$ and u_L are joined by a 2-rarefaction.
(iv) $\bar{\bar{u}}$ and u_L are joined by a 2-rarefaction.
(v) $\bar{\bar{u}}$ and u_L are joined by a 1-rarefaction.

(We assume here that $\lambda_2(u)$ increases from $\bar{u}$ to u_L. The case where $\lambda_2(u)$ decreases from $\bar{u}$ to u_L is discussed below.)

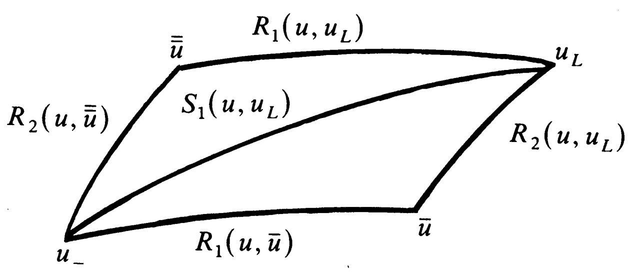

FIGURE 1

The system (2) with initial condition

$$u(x,0) = \begin{cases} u_R & \text{if } 0 < x < (s_1 - \lambda_1(u_R))T, \\ u_L & \text{otherwise} \end{cases},$$

has, for $t \leqslant T$, the solution u shown in Figure 2.

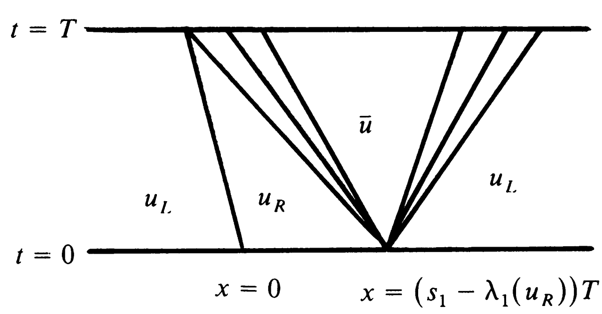

FIGURE 2. A line denotes a shock and a fan denotes a rarefaction.

Then

$$I_D(u, u_L; 0) = D(u_R, u_L)(s_1 - \lambda_1(u_R))T$$

and

$$I_D(u, u_L; T) = T\int_{\lambda_1(u_R)}^{\lambda_1(\bar{u})} D(\mu(\lambda), u_L)\, d\lambda + D(u_L, \bar{u})(\lambda_2(\bar{u}) - \lambda_1(\bar{u}))T$$
$$+ T\int_{\lambda_2(\bar{u})}^{\lambda_2(u_L)} D(\nu(\lambda), u_L)\, d\lambda,$$

where $\mu(\lambda)$ and $\nu(\lambda)$ denote parametrizations of $R_1(u, \bar{u})$ with respect to λ_1 and of $R_2(u, u_L)$ with respect to λ_2, respectively.

Now, with u_R, u_L, and $\bar{u}$ denoting the same states as in Figure 1, consider the following initial condition:

$$v(x, 0) = \begin{cases} u_L & \text{if } 0 < x < (\lambda_2(u_L) - s_1)T, \\ u_R & \text{otherwise.} \end{cases}$$

The solution v of this problem, for $t \leqslant T$, is given by the waves in Figure 3.

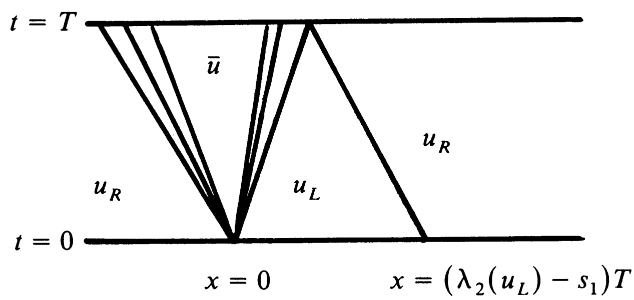

FIGURE 3

Then

$$I_D(v, u_R; 0) = D(u_L, u_R)(\lambda_2(u_L) - s_1)T$$

and

$$I_D(v, u_R; T) = T\int_{\lambda_1(u_R)}^{\lambda_1(\bar{u})} D(\mu(\lambda), u_R)\, d\lambda + T\int_{\lambda_2(\bar{u})}^{\lambda_2(u_L)} D(\nu(\lambda), u_R)\, d\lambda$$
$$+ D(\bar{u}, u_R)(\lambda_2(\bar{u}) - \lambda_1(\bar{u}))T.$$

To prove the theorem by contradiction, assume now that

(3) $$I_D(v, u_R; T) + I_D(u, u_L; T) \leqslant I_D(v, u_R; 0) + I_D(u, u_L; 0).$$

Thus

$$\int_{\lambda_1(u_R)}^{\lambda_1(\bar{u})} \{D(\mu(\lambda), u_L) + D(\mu(\lambda), u_R)\}\, d\lambda$$
$$+ \int_{\lambda_2(\bar{u})}^{\lambda_2(u_L)} \{D(\nu(\lambda), u_L) + D(\nu(\lambda), u_R)\}\, d\lambda$$
$$+ D(u_L, \bar{u})(\lambda_2(\bar{u}) - \lambda_1(\bar{u})) + D(\bar{u}, u_R)(\lambda_2(\bar{u}) - \lambda_1(\bar{u}))$$
$$\leqslant D(u_R, u_L)(s_1 - \lambda_1(u_R)) + D(u_L, u_R)(\lambda_2(u_L) - s_1).$$

Now, adding and subtracting

$$D(u_L, u_R)(\lambda_1(\bar{u}) - \lambda_1(u_R)) + D(u_L, u_R)(\lambda_2(u_L) - \lambda_2(\bar{u}))$$

we get

$$\begin{aligned} &\int_{\lambda_1(u_R)}^{\lambda_1(\bar{u})} \{D(\mu(\lambda), u_L) + D(\mu(\lambda), u_R) - D(u_L, u_R)\}\, d\lambda \\ &\quad + \int_{\lambda_2(\bar{u})}^{\lambda_2(u_L)} \{D(\nu(\lambda), u_L) + D(\nu(\lambda), u_R) - D(u_R, u_L)\}\, d\lambda \\ &\quad + (D(u_L, \bar{u}) + D(\bar{u}, u_R) - D(u_R, u_L))(\lambda_2(\bar{u}) - \lambda_1(\bar{u})) \leqslant 0. \end{aligned} \tag{4}$$

By the triangle inequality the two integrands and the third line above are positive. Since $\lambda_1(u_R) < \lambda(\bar{u})$ and $\lambda_2(\bar{u}) < \lambda_2(u_L)$, equality holds in (4) if and only if equality holds in each of the triangle inequalities, in particular only if

$$D(u_L, \bar{u}) + D(\bar{u}, u_R) = D(u_R, u_L). \tag{5}$$

A similar construction as above using the initial conditions

$$u_0(x) = \begin{cases} u_L & \text{if } 0 < x < \lambda_1(u_L) - s, \\ \bar{\bar{u}} & \text{otherwise,} \end{cases}$$

and

$$u_0'(x) = \begin{cases} \bar{\bar{u}} & \text{if } 0 < x < \lambda_2(v) - \lambda_1(v), \\ u_L & \text{otherwise,} \end{cases}$$

will, by the same argument, yield

$$D(u_R, \bar{\bar{u}}) + D(u_R, u_L) = D(\bar{\bar{u}}, u_L). \tag{6}$$

Now, (5) and (6) give

$$D(\bar{\bar{u}}, u_L) - D(\bar{u}, u_R) = D(u_L, \bar{u}) + D(u_R, \bar{\bar{u}}). \tag{7}$$

To see that there is no nondegenerate metric satisfying (7), consider the states u_R, u_L, $\bar{u}$, and $\bar{\bar{u}}$ in the space of Riemann invariants. In that space those states form a rectangle which in the metric D has side lengths $a \equiv D(u_L, \bar{\bar{u}})$, $b \equiv D(\bar{\bar{u}}, u_R)$, $c \equiv D(u_R, \bar{u})$, and $d \equiv D(\bar{u}, u_L)$ (see Figure 4).

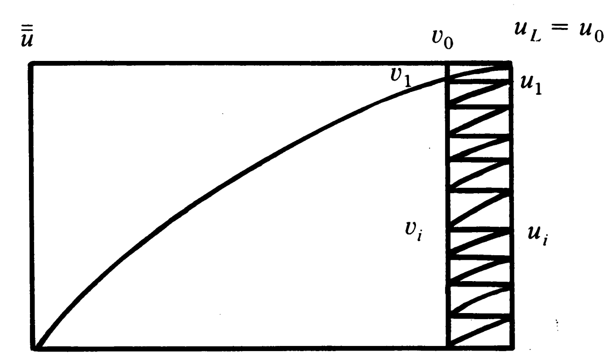

FIGURE 4

Then by (7)

$$a = b + c + d.$$

Now take states $\{v_i\}$ and $\{u_i\}$ ($u_0 = u_L$) joined by shock and rarefaction curves as in Figure 4 and let $\alpha_i \equiv D(v_i, u_i)$, $\beta_i \equiv D(v_{i+1}, v_i)$, $\gamma_i \equiv D(u_{i+1}, u_i)$.

Since by assumption shock and rarefaction curves do not coincide,

$$\sum_{i=0}^{N} \gamma_i = c < a$$

for some finite N.

Thus

$$\alpha_0 = \beta_0 + \gamma_0 + \alpha_1 = \sum_{i=0}^{N} (\beta_i + \gamma_i) + \alpha_{N+1}.$$

Now, if we take $\alpha_0 < c$,

$$D(u_{N+1}, v_{N+1}) = \alpha_{N+1} < -\sum_{i=0}^{N} \beta_i < 0$$

which is a contradiction to the fact that D is a positive metric. □

In the case where $\lambda_2(u)$ decreases from $\bar{u}$ to u_L a similar construction using the shock curves $S_2(u, \bar{\bar{u}})$ and $S_2(u, \hat{u})$ (see Figure 5) and the same initial value problems as above yield conditions similar to (5) and (6) from where the proof would proceed identically.

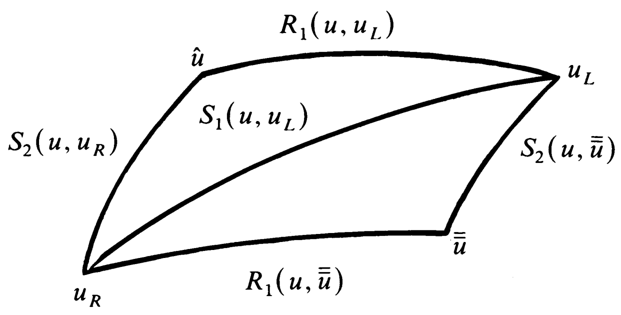

FIGURE 5

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