REVERSE HÖLDER INEQUALITIES FOR SPHERICAL HARMONICS

JAVIER DUOANDIKOETXEA

(Communicated by Richard R.Goldberg)

ABSTRACT. We prove that the $L^p$-norm with respect to the normalized Lebesgue measure on the sphere of any spherical harmonic of degree $k$ is bounded by a constant independent of the dimension times its $L^2$-norm. Several consequences are obtained from this result.

1. Introduction. In a previous paper [2] we were led to study the quotient $\|Y_k\|_p/\|Y_k\|_2$ where $Y_k$ stands for a spherical harmonic of degree $k$ in $\mathbb{R}^n$ and the $L^p$-norms are taken with respect to the normalized Lebesgue measure on the sphere $S^{n-1}$, i.e.,

$$\|Y_k\|_p = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |Y_k(u)|^p \, d\sigma(u)$$

$(d\sigma(u)) =$Lebesgue measure on $S^{n-1}$, $|S^{n-1}| =$measure of $S^{n-1} = 2\pi^{n/2}\Gamma(n/2)^{-1}$.

When $p > 2$, Hölder's inequality provides the trivial lower bound 1. We prove here, by using two different approaches, that we have the upper bound $(p - 1)^{1/2}$, independent of $n$. The first proof is the same as in [2] with further precision; the second uses two well-known but far from trivial facts: the Bochner-Hecke formula and the Beckner-Hausdorff-Young inequality.

For $p < 2$, Hölder’s inequality gives the upper bound 1 and a lower bound independent of $n$ can be found by using the preceding part and interpolation.

Similar results can be obtained for any polynomial of degree $k$ and any sphere in $\mathbb{R}^n$ using the decomposition of the polynomial restricted to the sphere as a sum of spherical harmonics. Some other consequences are also given.

We use the same notation for the spherical harmonic $Y_k$ defined on $S^{n-1}$ and the solid harmonic defined in all $\mathbb{R}^n$ which are related by $Y_k(x) = Y_k(x/|x|)|x|^k$.

We are indebted to José L. Rubio de Francia for suggesting certain applications and improvements to this paper.

2. The main theorem and its two proofs.

THEOREM 1. Let $Y_k$ be a spherical harmonic of degree $k$ in $\mathbb{R}^n$. Then, if $p \geq 2$,

$$\|Y_k\|_p \leq (p - 1)^{1/2}\|Y_k\|_2.$$

PROOF. We use induction on $k$. Let $k = 1$. By rotation it is enough to prove the theorem for $Y_1(u) = u_1$. But a simple computation gives

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} |u_1|^p \, d\sigma(u) = \frac{\Gamma(n/2)\Gamma((p + 1)/2)}{\pi^{1/2}\Gamma((n + p)/2)}$$

Received by the editors July 14, 1986.

1980 Mathematics Subject Classification (1985 Revision). Primary 42C05.

©1987 American Mathematical Society

0002-9939/87 $1.00 + .25 per page
and \(\|Y_1\|_2 = n^{-1/2}\); then it follows from the properties of the gamma function that

\[\|Y_1\|_p / \|Y_1\|_2 \leq (p - 1)^{1/2}\]

Assume now the theorem for \(k - 1\) and let \(Y_k\) be a spherical harmonic of degree \(k\). We claim that it is enough to obtain

\[\|Y_k\|_p \leq \left( \frac{p - 1}{k(kp + n - 2)} \right)^{1/2} \|\nabla Y_k\|_p \quad \text{with equality for } p = 2.\]

In fact, the induction hypothesis and Minkowski’s inequality imply

\[\|\nabla Y_k\|_p \leq (p - 1)^{(k-1)/2} \|\nabla Y_k\|_2\]

and then using (1) successively with \(L^p\) - and \(L^2\)-norms, we get

\[\|Y_k\|_p \leq \left( \frac{p - 1}{k(kp + n - 2)} \right)^{1/2} (p - 1)^{(k-1)/2} \|\nabla Y_k\|_2 \]

\[= (p - 1)^{k/2} \left( \frac{2k + n - 2}{kp + n - 2} \right)^{1/2} \|Y_k\|_2 \leq (p - 1)^{k/2} \|Y_k\|_2.\]

Finally we prove (1). From the homogeneity of solid harmonic \(Y_k\) and Green’s formula we have

\[\int_{S^{n-1}} |Y_k|^p d\sigma = \frac{1}{k} \int_{S^{n-1}} |Y_k|^{p-1} \text{sgn } Y_k \frac{\partial Y_k}{\partial \nu} d\sigma\]

\[= \frac{1}{k} \int_{|x|<1} \nabla(|Y_k|^{p-1} \text{sgn } Y_k) \nabla Y_k \, dx\]

\[= \frac{p - 1}{k} \int_{|x|<1} |Y_k|^{p-2} |\nabla Y_k|^2 \, dx\]

\[= \frac{p - 1}{k(kp + n - 2)} \int_{S^{n-1}} |Y_k|^{p-2} |\nabla Y_k|^2 \, d\sigma.\]

When \(p = 2\) we get the equality in (1); for \(p > 2\), just apply Hölder’s inequality with exponents \(p/(p - 2)\) and \(p/2\). \(\square\)

The second way to prove Theorem 1 gives a somewhat more precise result:

**Theorem 1'.** Let \(Y_k\) be a spherical harmonic of degree \(k\), \(2 \leq p < \infty\) and \(1/p + 1/q = 1\). Then,

\[\|Y_k\|_p \leq (p - 1)^{k/2} \|Y_k\|_q.\]

**Proof.** Bochner-Hecke’s formula states that

\[\mathcal{F}(e^{-\pi|x|^2} Y_k(x)) = i^{-k} e^{-\pi|x|^2} Y_k(\xi)\]

(\(\mathcal{F}\) is the Fourier transform and \(Y_k\) is here the solid harmonic; see Stein [4, p. 71]). The Hausdorff-Young inequality can be written in the form

\[\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^n)} \leq (q^{1/q} / p^{1/p})^{n/2} \|f\|_{L^p(\mathbb{R}^n)}\]

(see Beckner [1]). When this inequality is applied to (2), taking into account that

\[\|e^{-\pi|x|^2} Y_k(x)\|_{L^p(\mathbb{R}^n)} = \frac{\Gamma((kp + n)/2)}{\pi^{k/2} 2^{kp+n}/2 \Gamma(n/2)} \|Y_k\|_p\]
we get

\[ \| Y_k \|_p \leq \left( \frac{p}{q} \right)^{k/2} \frac{\Gamma((kq + n)/2)^{1/q} \Gamma(n/2)^{1/p}}{\Gamma((kp + n)/2)^{1/p} \Gamma(n/2)^{1/q}}. \]

Since \( \log \Gamma \) is a convex function in \((0, \infty)\), the last factor is \( \leq 1 \) and the theorem is proved. \( \Box \)

The function \( Y_k(x) = (x_1 + ix_2)^k \) is harmonic and homogeneous of degree \( k \). If we compute the \( L^p \)-norm of its restriction to \( S^{n-1} \) we see that the preceding results are sharp in the following sense: for constants independent of \( n \), no exponent less than \( k/2 \) can appear in the right-hand side of Theorems 1 and 1'; in fact

\[ \sup_n \frac{\| Y_k \|_p}{\| Y_k \|_2} \leq c(k)p^{k/2}. \]

3. Some consequences. (a) Theorem 1 and the method of rotations give the following: Let \( \{ Y_j \} \) be a basis of the linear space of spherical harmonics of degree \( k \) in \( \mathbb{R}^n \) and \( d(k, n) \) its dimension. If we normalize the \( Y_j \) in such a way that \( \| Y_j \|_2 = d(k, n)^{-1/2} \) and define the operators \( (R_j f)(\xi) = Y_j(\xi/|\xi|)\hat{f}(\xi) \), there then exists \( C_{p, k} \) independent of \( n \) such that

\[ \left\| \left( \sum_{j=1}^{d(k, n)} |R_j f|^2 \right)^{1/2} \right\|_p \leq C_{p, k} \| f \|_p, \quad 1 < p < \infty, \]

with \( C_{p, k} = O(p^{1+k/2}) \), \( p \to \infty \), and \( = O((p - 1)^{-1}) \), \( p \to 1 \). This was our motivation for Theorem 1 and can be seen in [2].

(b) For \( p < 2 \), a lower bound for \( \| Y_k \|_p/\| Y_k \|_p \) can be obtained from Theorem 1, namely

COROLLARY 2. If \( 0 < p < 2 \),

\[ \| Y_k \|_2 \leq e^{k((2/p) - 1)} \| Y_k \|_p. \]

PROOF. Let \( s > 2 \). By interpolation and Theorem 1

\[ \| Y_k \|_2 \leq \| Y_k \|_p \| Y_k \|_s^{1-s} \leq (s - 1)(1-\theta)^{k/2} \| Y_k \|_p \| Y_k \|_2^{1-\theta} \]

with \( 1/2 = \theta/p + (1 - \theta)/s \). Then,

\[ \| Y_k \|_2 \leq (s - 1)(1-\theta)^{k/2} \| Y_k \|_p \]

and the corollary follows from

\[ \inf_{s > 2} (s - 1)^{(1-\theta)/\theta} = \lim_{s \to 2} (s - 1)^{(1-\theta)/\theta} = e^{2(p-1)}. \]  \( \Box \)

(c) Theorem 1 and Corollary 2 have the following similar versions in the case of arbitrary polynomials.

COROLLARY 3. Let \( P_k \) be any polynomial of degree \( k \) and \( S \) any sphere in \( \mathbb{R}^n \). Then, if \( 2 < p < \infty \),

\[ \left( \frac{1}{|S|} \int_S |P_k|^p \, d\sigma \right)^{1/p} \leq p^{k/2} \left( \frac{1}{|S|} \int_S |P_k|^2 \, d\sigma \right)^{1/2}. \]
and if \(0 < p < 2\),
\[
\left( \frac{1}{|S|} \int_S |P_k|^2 \, d\sigma \right)^{1/2} \leq 4^{k(2/p-1)} \left( \frac{1}{|S|} \int_S |P_k|^p \, d\sigma \right)^{1/p}.
\]

**Proof.** Since translation and dilation change a polynomial into another of the same degree, it will be enough to prove the result for \(S = S^{n-1}\). But on \(S^{n-1}\), \(P_k = \sum_{j=0}^{k} Y_j\) where \(Y_j\) is a spherical harmonic of degree \(j\). By Theorem 1,
\[
\|P_k\|_p \leq \sum_{j=0}^{k} \|Y_j\|_p \leq \sum_{j=0}^{k} (p - 1)^{j/2} \|Y_j\|_2
\]
and the first part of the corollary is a consequence of the orthogonality of the \(Y_j\) after applying Cauchy-Schwarz inequality.

The second part follows from the first as in the proof of Corollary 2. \(\square\)

(d) The size of the constants in Theorem 1 and Corollary 3 makes it possible to give an estimate of exponential type with constant independent of \(n\).

**Corollary 4.** Let \(P_k\) be a polynomial of degree \(k\) in \(\mathbb{R}^n\). Then, for any sphere \(S\) in \(\mathbb{R}^n\)
\[
\frac{1}{|S|} \int_S \exp \left[ \frac{P_k(u)}{\|P_k\|_2} \right] \, d\sigma(u) \leq C(k, \lambda)
\]
with a constant independent of \(n\) if \(\lambda < 2/k\) (also for \(\lambda = 2/k\) if \(k \geq 6\)).

(e) The following application is based on a result in probability theory.

Let \(Y^{(n)} = (Y_1, \ldots, Y_n, 0, 0, \ldots)\), \(n = 1, 2, \ldots\), be random variables such that \((Y_1, \ldots, Y_n)\) are uniformly distributed in a sphere of \(\mathbb{R}^n\) of radius \(r_n = (n/2\pi)^{1/2}\) and let \(X = (X_1, \ldots, X_m, \ldots)\) be a random variable with \(X_1, \ldots, X_m, \ldots\) independent and having \(\exp(-\pi|x|^2)\) as distribution function. Then, the sequence \(Y^{(n)}\) converges to \(X\) in law.

This result is known and can be easily proved. If \(E_n(\ell)\) stands for the expectation of \(\ell\) with respect to \(Y^{(n)}\) and \(E(\ell)\) is the expectation with respect to \(X\), we have
\[
\lim_{n \to \infty} E_n(\ell) = E(\ell).
\]
Taking now as \(\ell\) the \(p\)th power of a polynomial of degree \(k\) and using Corollary 3 we have

**Corollary 5.** Let \(X = (X_1, \ldots, X_n, \ldots)\) be a random variable where the \(X_i\) are independent and have \(\exp(-\pi|x|^2)\) as distribution function. If \(P(X)\) is a polynomial of degree \(k\) in the variables \(X_1, \ldots, X_n, \ldots\) the following reverse Hölder inequalities hold.
\[
E(|P|^{p})^{1/p} \leq p^{k/p} E(|P|^2)^{1/2}, \quad 2 < p < \infty,
\]
\[
E(|P|^2)^{1/2} \leq 4^{k(2/p-1)} E(|P|^p)^{1/p}, \quad 0 < p < 2.
\]

(f) Upper bounds for the quotient \(\|Y_k\|_p/\|Y_k\|_2\), \(2 < p < \infty\), are interesting also in the study of Bochner-Riesz operators on the sphere. In that case the dimension \(n\) of the underlying space is kept fixed and the interest is in the behaviour of the quotient for \(k \to \infty\). Sharp bounds have been obtained by C. Sogge [3].

\[
\|Y_k\|_p/\|Y_k\|_2 = O(k^\alpha(p))
\]
where
\[ \alpha(p) = \frac{(n - 2)(p - 2)}{4p} \quad \text{if} \quad 2 \leq p \leq 2n/(n - 2), \]
\[ = \frac{(n - 2)}{2} - \frac{(n - 1)}{p} \quad \text{if} \quad 2n/(n - 2) < p \leq \infty. \]

From the proof of Theorem 1' and more precisely from inequality (3) it follows immediately that
\[ \|Y_k\|_p/\|Y_k\|_q = O(k^{(n-1)(p-2)/2p}). \]

But, if we put \(\|Y_k\|_2\) instead of \(\|Y_k\|_q\) the bound we get for the quotient is not sharp and it does not apply to obtaining nontrivial results for Bochner-Riesz operators.

Using (4) we can get in a trivial way the bound \(O(k^{2\alpha(p)})\) for the quotient \(\|Y_k\|_p/\|Y_k\|_q\). It is easily verified that \(2\alpha(p)\) is less than our exponent when \(p\) is close to 2 and bigger when \(p\) is close to \(\infty\) (for \(n > 3\)).

REFERENCES


División de Matemáticas, Universidad Autónoma, 28049 Madrid, Spain