PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 3, November 1987

DOUBLY-PERIODIC SOLUTIONS OF A FORCED SEMILINEAR WAVE EQUATION

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(Communicated by Walter Littman)

ABSTRACT. Existence results are obtained for doubly-periodic solutions of a semilinear wave equation when the nonlinearity is bounded in one side.

In this work we study the existence of weak doubly 2π-periodic solutions of the semilinear wave equation

$$u_{tt} - u_{xx} + g(u) = f(t,x), \qquad (t,x) \in R^2, \tag{1}$$

where f is a given 2π-periodic function in t and x, g is a continuous function, and we assume, among other conditions to be specified later, the following:

$$g \text{ is nondecreasing and } g(-\infty) > -\infty, \quad g(+\infty) = +\infty. \tag{2}$$

When one looks for periodic spatially homogeneous (i.e. independent of x) solutions of (1), then one is lead to the periodic problem for the O.D.E.

$$\frac{d^2u}{dt^2} + g(u) = f(t).$$

It is known that this last problem admits a solution when g satisfies (2) and $(1/2\pi)\int_0^{2\pi} f > g(-\infty)$. (See for example [**5**, **7**].) The motivation of this paper is to extend in a certain sense this result to equation (1).

The Dirichlet-periodic boundary value problem for (1) has been extensively studied by Bahri and Brezis [**1**] (see also [**2**, **3**]). Their corresponding condition on g is

$$g \text{ is nondecreasing and } |g(u)| \leq \gamma|u| + c, \quad u \in R, \tag{3}$$

where γ and c are constants with $\gamma < |\lambda_{-1}|$. Here λ_{-1} is the first negative eigenvalue of the linear operator $\Box = \partial^2/\partial t^2 - \partial^2/\partial x^2$ when it acts on functions sastisfying the boundary conditions. Hypothesis (3) does not allow the crossing of g and the eigenvalues of $\Box$ different from $\lambda_0 = 0$. Also, it is obvious that that the growth of g must be of linear type at most. The results in [**1**] can be easily translated to the doubly-periodic case.

In contrast with (3), our condition on g is of a different nature. Actually, when (2) is verified, g may cross other eigenvalues besides λ_0 or grow arbitrarily in the

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Received by the editors July 18, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35L70, 35L05.

Key words and phrases. Nonlinear wave equation, doubly periodic solutions, resonance, L^∞-bounds.

The authors wish to acknowledge the support from CAICYT, Ministerio de Educación y Ciencia, Spain.

positive direction. On the other side g must be bounded in the negative axis, which is not required by (3).

We should also mention the paper of Ward [**8**] on the doubly-periodic problem for (1). The results in [**8**] are related to ours, although g is not allowed to interact with λ_0.

Further discussions on the connections with those works will be given at the end of the paper.

Our method of proof is based on a simple idea. We consider a sequence of truncated equations such that it is possible to apply the results in [**1**] to each one of them. Then we obtain a uniform L^∞-bound on the solutions of the corresponding truncated problems; implying therefore that the solution of some of these problems is, at the same time, a solution of the original one. The technique used to get the bounds is based on [**1**] combined with some additional estimates similar to those in [**8**].

PRELIMINARIES. Denote by H the Hilbert space $L^2(J)$, $J = (0,2\pi)\times(0,2\pi)$, with inner product $(\cdot,\cdot)$. Throughout the paper, a function on J will be identified, whenever needed, to its doubly-periodic extension to R^2.

The realization in H of the wave operator with periodic conditions, denoted by A, is defined as follows. Let $\mathcal{D}$ be the class of test functions $\varphi \in C^2(\overline{J})$ verifying

$$\varphi(t,0)-\varphi(t,2\pi)=\varphi_x(t,0)-\varphi_x(t,2\pi)=0,$$
$$\varphi(0,x)-\varphi(2\pi,x)=\varphi_t(0,x)-\varphi_t(2\pi,x)=0, \qquad t,x\in[0,2\pi].$$

A is given by

$$D(A)=\left\{u\in H/\varphi\in\mathcal{D}\to\int_J u\Box\varphi \text{ is } L^2\text{-continuous}\right\},$$
$$(u,\Box\varphi)=(Au,\varphi)\quad\text{for every } u\in D(A),\quad \varphi\in\mathcal{D}.$$

It is known that A is a selfadjoint unbounded linear operator in H with closed range and $\lambda_0=0$ is an eigenvalue of A of infinite multiplicity. The kernel and the range of A are explicitly given by

$$N(A)=\Big\{u_0\in H/u_0(t,x)=\bar u_0+p(t+x)+q(t-x)\text{ a.e. } J,$$
$$\bar u_0\in R;\ p,q\in L^2_{\text{loc}}(R),\ 2\pi\text{-periodic and }\int_0^{2\pi}p=\int_0^{2\pi}q=0\Big\},$$
$$R(A)=N(A)^\perp=\Big\{u_1\in H/\int_0^{2\pi}u_1(t-x,x)\,dx$$
$$=\int_0^{2\pi}u_1(t+x,x)\,dx=0,\text{a.e. } t\in(0,2\pi)\Big\},$$

and the natural projection onto the kernel,

$$Pu(t,x)=\bar u+p(t+x)+q(t-x),\qquad u\in H,$$

where

$$\bar u=\frac{1}{(2\pi)^2}\int_J u,\ p(t)=\frac{1}{2\pi}\int_0^{2\pi}u(t-x,x)\,dx-\bar u,$$
$$q(t)=\frac{1}{2\pi}\int_0^{2\pi}u(t+x,x)\,dx-\bar u. \tag{4}$$

We shall need the following regularity result (see [**4**]). Given $f \in R(A)$, let $u \in R(A)$ be the unique solution of $Au = f$ in $R(A)$. Then

$$u \in L^\infty(J), \quad \|u\|_{L^\infty} \le c\|f\|_{L_1} \tag{5}$$

for some constant c independent of u and f.

By a weak solution of the doubly 2π-periodic problem or (1) we understand a function $u \in L^\infty(J)$ such that

$$(u, \Box\varphi) + (g(u), \varphi) = (f, \varphi) \quad \text{for every } \varphi \in \mathcal{D}. \tag{6}$$

Clearly, $u \in L^\infty(J)$ verified (6) if and only if u verifies

$$Au + g(u) = f. \tag{7}$$

We can now state our result.

THEOREM. *Assume that g satisfies* (2) *and the following condition is verified:*

(8) *$f \in L^\infty(J)$ admits a decomposition in the form $f = f^* + f^{**}$ with $f^* \in R(A) \cap L^\infty(J)$, $f^{**}(t,x) \ge g(-\infty) + \delta$ a.e. $(t,x) \in J$ for some $\delta > 0$. Then there exists at least one solution of* (7) *in $L^\infty(J)$.*

In the proof of the theorem we shall need a preliminary result that can be proved following the lines of [**1**].

LEMMA. *Assume*

(i) *g is a bounded nondecreasing continuous function.*

(ii) *$f \in L^\infty(J)$ admits a decomposition in the form $f = f^* + f^{**}$ with $f^* \in R(A)$, $g(+\infty) - \delta \ge f^{**}(t,x) \ge g(-\infty) + \delta$ a.e. $(t,x) \in J$ for some $\delta > 0$. Then there exists at least one solution of* (7) *in $L^\infty(J)$.*

PROOF OF THE THEOREM. It is not restrictive to assume $g(-\infty) = 0$. In consequence $g \ge 0$ over the entire real line. Let us consider the sequence of truncated functions

$$g_n(u) = \min[g(u), g(n)], \qquad n = 1, 2, \ldots,$$

and the corresponding problems

$$Au + g_n(u) = f. \tag{9}$$

It is clear that g_n verifies condition (i) of the previous lemma and, since $g_n(-\infty) = g(-\infty) = 0$, and $g_n(+\infty) = g(n)$, (ii) is verified for n sufficiently large. Hence, for large n, (9) admits a solution $u_n \in L^\infty(J)$. We will conclude the proof by showing the existence of an L^∞-estimate of u_n independent of n, implying therefore that $g_n(u_n) = g(u_n)$ and u_n is a solution of (7) for large n.

We denote by C_i, $i = 1, 2, \ldots$, positive constants independent of n. Each u_n can be decomposed as $u_n = u_{0n} + u_{1n}$ with $u_{0n} = Pu_n$, $u_{1n} = (I - P)u_n$.

Since constant functions belong to $N(A)$ and $g_n(u_n) - f \in R(A)$, one has $(g_n(u_n) - f, 1) = 0$. Now,

$$\|g_n(u_n)\|_{L^1} = \int_J g_n(u_n) = \int_J f \qquad (g_n \ge 0)$$

and

$$\|Au_n\|_{L^1} = \|g_n(u_n) - f\|_{L^1} \le \int_J f + \|f\|_{L^1}.$$

From (5) we obtain

$$\|u_{1n}\|_{L^\infty} \le C_1. \tag{10}$$

By (2) and (8) one can find positive constants γ and k such that

$$[g_n(u+\xi) - f^{**}(t,x)]u \ge \gamma|u| - k \tag{11}$$

for $u \in R$, $|\xi| \le C_1$, $(t,x) \in J$, and large n.

Using the facts that $u_{0n} \in N(A)$ and $g_n(u_n) - f^{**} \in R(A)$, and applying (11) with $u = u_{0n}$, $\xi = u_{1n}$, one gets

$$\gamma \int_J |u_{0n}| - (2\pi)^2 k \le (g_n(u_{0n} + u_{1n}) - f^{**}, u_{0n}) = 0,$$

from where

$$\|u_{0n}\|_{L^1} \le C_2. \tag{12}$$

We can now write u_{0n} as

$$u_{0n}(t,x) = \bar{u}_{0n} + p_n(t+x) + q_n(t-x)$$

with p_n, q_n essentially bounded, 2π-periodic and with mean value zero. The relations (4) together with (12) imply

$$|\bar{u}_{0n}|, \|p_n\|_{L^1}, \|q_n\|_{L^1} \le C_3. \tag{13}$$

Therefore it is enough to find L^∞-estimates for p_n and q_n.

Since $g_n(u_n) - f^{**} \in R(A)$,

$$\int_0^{2\pi} g_n(u_n(t-x,x))\,dx = \int_0^{2\pi} f^{**}(t-x,x)\,dx, \tag{14}$$

$$\int_0^{2\pi} g_n(u_n(t+x,x))\,dx = \int_0^{2\pi} f^{**}(t+x,x)\,dx \tag{15}$$

a.e. $t \in (0, 2\pi)$. From the previous estimates,

$$u_n(t,x) \ge -C_4 + p_n(t+x) + q_n/(t-x) \quad \text{a.e. } J,\ C_4 = C_1 + C_3,$$

and (14) together with the monotonicity of g_n imply

$$\int_0^{2\pi} g_n(-C_4 + p_n(t) + q_n(x))\,dx \le 2\pi \|f^{**}\|_{L^\infty}. \tag{16}$$

Let $M_n = \text{ess sup}_{(0,2\pi)}\, p_n$ ($M_n \ge 0$ because $\int_0^{2\pi} p_n = 0$) and

$$\Sigma_n = \{x \in (0,2\pi) : |q_n(x)| \ge M_n/2\}.$$

By (13) $\text{meas}(\Sigma_n) \le 2C_3/M_n$, and by (16)

$$\begin{aligned} 2\pi\|f^{**}\|_{L^\infty} &\ge \int_{\Sigma_n} g_n + \int_{[0,2\pi]-\Sigma_n} g_n \\ &\ge g_n\left(-C_4 + p_n(t) - \frac{M_n}{2}\right) \cdot \left(2\pi - \frac{2C_3}{M_n}\right) \quad \text{a.e. } t \in (0,2\pi). \end{aligned}$$

It follows

$$\min\{g(n), g(-C_4 + M_n/2)\}(2\pi - 2C_3/M_n) \le 2\pi\|f^{**}\|_{L^\infty},$$

which forces the boundedness of M_n.

Changing the roles of p_n and q_n, starting with (15), and repeating the process one also find upper bounds for q_n. Say $p_n(t),\ q_n(t) \le C_5$ a.e. t. Going back to (14) and using the estimate

$$u_n(t,x) \le C_4 + p_n(t+x) + C_5$$

one obtains

$$g_n(C_4 + C_5 + p_n(t)) \ge \delta > 0 \quad \text{a.e. } t.$$

Let $N_n = \operatorname{ess\,inf}_{(0,2\pi)} p_n$ $(N_n \le 0)$. Then

$$g_n(C_4 + C_5 + N_n) \ge \delta > 0,$$

which implies the boundedness of N_n, since $g(-\infty) = 0$. A similar reasoning for q_n using (15) finishes the proof.

REMARKS. 1. Hypothesis (2) of the theorem can be generalized in the following sense: g is monotone and exactly one of the limits $g(+\infty), g(-\infty)$ is finite. (Of course (8) may need a slight change.)

2. Hypothesis (8) was first formulated in [**1**, **2**] and, as mentioned there in a similar context, is a sharp condition for the solvability of (7). In fact, when $g(-\infty) < g(u)$ for all $u \in R$ (and this is the case for an increasing g) it is easily seen that (8) characterizes the solvability of (7). However, when $g(-\infty) = g(u)$, $u \le c$ for some c, (8) is only sufficient. A necessary condition for the solvability is

> $f \in L^\infty(J)$ admits a decomposition in the form $f^* + f^{**}$ with $f^* \in R(A) \cap L^\infty(J)$, $f^{**}(t,x) \ge g(-\infty)$ a.e. $(t,x) \in J$.

We do not know whether this last condition is also sufficient in this case or not.

3. Some model nonlinearities verifying (2) are αu^+ $(\alpha \le 1), u^2u^+, e^u, \ldots$. The results in [**1**, **8**] do not apply to these examples. On the other hand we cannot study a nonlinear term of the type $\alpha u^+ - \beta u^-$ $(0 < \alpha, \beta < 1)$ that can be studied from the results in [**1** or **8**].

4. A key factor in our proof is the existence of positive functions in the kernel of A. Therefore it is not possible to adapt the proof to a Dirichlet-periodic problem (DP) for equation (1) of the type studied in [**1**]. Assuming that the (DP) problem is posed over $(0,2\pi)\times(0,\pi)$ the function $\phi_1(t,x) = \sin x$ belongs to $N(A-\lambda_1 I)$, $\lambda_1 = 1$. It seems possible, however, to obtain some results when the resonance is at λ_1 by using the positivity of ϕ_1.

5. Apparently, the obtaining of bounds of the solution relies very heavily on the specific structure of $N(A), A = \Box$. It would be of interest to obtain results of a similar flavor for other semilinear problems with an infinite-dimensional kernel. A good example might be the problem induced by the beam equation. A representation of $N(A)$, $A = \partial^2/\partial t^2 + \partial^4/\partial x^4$, may be seen in [**6**].

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