

SMALL PERTURBATIONS AND THE EIGENVALUES OF THE LAPLACIAN ON LARGE BOUNDED DOMAINS

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ABSTRACT. Denote by Δ_L^D the Laplacian on a hypercube in \mathbf{R}^d with side length πL . Also denote by $N(\lambda, A)$ the number of eigenvalues of the operator A below λ . If $V \geq 0$ is a bounded function of compact support, ($V > 0$ on a set of positive measure) then $N(-\Delta_L^D, \lambda) - N(-\Delta_L^D + V, \lambda)$ is not bounded as $L \rightarrow \infty$ for dimension $d > 1$.

1. Introduction. Let C_L be the hypercube in \mathbf{R}^d for $d > 1$ with side length πL centered at the origin. Denote by Δ_L^D the Laplacian on C_L with Dirichlet boundary conditions on ∂C_L . Let V be a bounded nonnegative potential with support in C_1 and set $H_0^{(L)} = -\Delta_L^D$, $H_1^{(L)} = -\Delta_L^D + V$.

If H is a selfadjoint operator bounded below with purely discrete spectrum we denote by $E_n(H)$, $n = 0, 1, \dots$, the eigenvalues of H put in increasing order and counted according to their multiplicity. We set $N(H, \lambda) = \#\{n \mid E_n(H) \leq \lambda\}$. The main object of investigation in this note is the function

$$(1) \quad \varphi_L(\lambda) = N(H_0^{(L)}, \lambda) - N(H_1^{(L)}, \lambda).$$

For any L $\varphi_L(\lambda)$ is a nonnegative function.

We ask whether $\varphi_L(\lambda)$ remains bounded for fixed λ if L goes to infinity. Our intuition (and the one of many experts I asked) suggests that φ_L should be bounded since the perturbation V is independent of L and so the "effect of V " should not grow with L but should rather become smaller. However, this intuition is erroneous, as we shall prove below.

As a corollary we obtain that also

$$(2) \quad \tilde{\varphi}_L(\lambda) = N(-\Delta_{C_L - C_1}^{D,N}) - N(-\Delta_{C_L - C_1}^{D,D})$$

is unbounded where $-\Delta_{C_L - C_1}^{D,N}$ (resp. $-\Delta_{C_L - C_1}^{D,D}$) is $-\Delta$ on $L^2(C_L - C_1)$ with Dirichlet boundary conditions on ∂C_L and Neumann (resp. Dirichlet) boundary conditions on ∂C_1 . The unboundedness of $\tilde{\varphi}_L$ is perhaps even less intuitive than that of φ_L .

Our proof that φ_L is unbounded as L increases relies heavily (though not obviously) on the high symmetry of the Laplacian, as was pointed out to us by H. Siedentop. It would be interesting to know the behavior of φ_L for $-\Delta$ replaced by some $H = -\Delta + W$. However, our method does not work in that case.

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2. The result. We are ready to state our result. Assume that $V(x) > 0$ on a set of positive Lebesgue measure.

THEOREM 1. (i) For any $\lambda > 0$ we have

$$\sup_{L \in \mathbf{R}} \varphi_L(\lambda) = \infty.$$

(ii) For a dense set $E \subset [0, \infty)$ and any $\lambda \in E$ we have

$$\sup_{L \in \mathbf{N}} \varphi_L(\lambda) = \infty.$$

3. Proof of (i). To prove this result we consider first the eigenvalues of the Laplacian $-\Delta_L^D$ on C_L . We denote by \mathbf{Z}_{++} the strictly positive integers. For $\mathbf{n} \in (\mathbf{Z}_{++})^d$ we set $|\mathbf{n}| = \left(\sum_{i=1}^d n_i^2\right)^{1/2}$. The eigenvalues of $-\Delta_L^D$ are given by

$$(3) \quad E_{\mathbf{n}}^{(L)} = \frac{1}{L^2} \sum_{i=1}^d n_i^2 \quad \text{for } \mathbf{n} \in (\mathbf{Z}_{++})^d.$$

The multiplicity of $E_{\mathbf{n}}^{(L)}$ equals the number of points $\mathbf{m} \in (\mathbf{Z}_{++})^d$ with $|\mathbf{m}| = |\mathbf{n}|$ (see Reed-Simon [2]). The first observation is

PROPOSITION 1. For any M there is an $\mathbf{n} \in (\mathbf{Z}_{++})^d$ such that $|\mathbf{m}| = |\mathbf{n}|$ has at least M distinct solutions $\mathbf{m} \in (\mathbf{Z}_{++})^d$.

PROOF. It is enough to prove the result for $d = 2$. In this case the number $l(a)$, $a \in \mathbf{N}$, of solutions $m \in \mathbf{Z}^2$ of $|m|^2 = a$ is exactly known by the following theorem of Gauss.

THEOREM (GAUSS). If

$$(4) \quad a = 2^n p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r} q_1^{m_1} q_2^{m_2} \cdots q_s^{m_s}$$

is the decomposition of a into prime factors with $p_i \equiv 1 \pmod{4}$ and $q_i \equiv 3 \pmod{4}$, the number $l(a)$ of distinct solutions of $|m|^2 = a$ with $m \in \mathbf{Z}^2$ is given by

$$l(a) = \begin{cases} 4(n_1 + 1)(n_2 + 1) \cdots (n_r + 1) & \text{if } m_i \equiv 0 \pmod{2} \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

For a proof we refer to Fricker [1, Satz 1 in §2].

From this theorem the assertion of the proposition follows. \square

COROLLARY. For any $M, L_0, E_0 > 0$ there is an $L > L_0$ such that E_0 is an eigenvalue of $H_0^{(L)}$ of multiplicity at least M .

PROOF. By Proposition 1 there is a number $\alpha > E_0 L_0^2$ such that $|\mathbf{m}|^2 = \alpha$ has at least M solutions $\mathbf{m} \in (\mathbf{Z}_{++})^d$. So with $L := (\alpha/E_0)^{1/2} > L_0$ we have

$$E_0 = \frac{1}{L^2} E_{\mathbf{m}}^{(1)} = E_{\mathbf{m}}^{(L)}.$$

Moreover $E_{\mathbf{m}}^{(L)}$ has multiplicity at least M . \square

The idea of the proof is that all the eigenvalues of $H_1^{(L)}$ are strictly bigger than the corresponding eigenvalues of $H_0^{(L)}$. So, if $\lambda = E_{\mathbf{m}}^{(L)}$ has multiplicity at least M the function satisfies $\varphi_L(\lambda) \geq M$.

Thus the proof of (i) is done once we know

PROPOSITION 2. For any L and $n \in \mathbf{N}$

$$E_n(H_1^{(L)}) > E_n(H_0^{(L)}).$$

PROOF. Suppose that f_0, \dots, f_m is an orthonormal set of eigenfunctions of $H_0^{(L)}$ spanning Rank $P_{(-\infty, E_n(H_0^{(L)}))}$ where P denotes the spectral projection of $H_1^{(L)}$. Denote by D the common domain of the operators $H_0^{(L)}$ and $H_1^{(L)}$. By the min-max-principle (see Reed-Simon [2]) we have

$$\begin{aligned}
 E_n(H_0^{(L)}) &= \inf_{\substack{\varphi \in D, |\varphi|=1 \\ \varphi \perp f_0, \dots, f_m}} \langle \varphi, H_0^{(L)} \varphi \rangle \\
 (*) \quad &\leq \inf_{\substack{\varphi \in D, |\varphi|=1 \\ \varphi \perp f_0, \dots, f_m}} \langle \varphi, H_1^{(L)} \varphi \rangle \leq E_n(H_1^{(L)}).
 \end{aligned}$$

It is a standard compactness argument to show that the infimum

$$\inf_{\substack{\varphi \in D, |\varphi|=1 \\ \varphi \perp f_0, \dots, f_m}} \langle \varphi, H_1^{(L)} \varphi \rangle$$

is attained by some function $\varphi_0 \in D$, $|\varphi_0| = 1$.

Now, suppose that equality holds in (*). Then

$$\langle \varphi_0, H_1^{(L)} \varphi_0 \rangle = \langle \varphi_0, H_0^{(L)} \varphi_0 \rangle = \inf_{\substack{\varphi \in D, |\varphi|=1 \\ \varphi \perp f_0, \dots, f_m}} \langle \varphi, H_0^{(L)} \varphi \rangle$$

and consequently φ_0 is an eigenfunction of $H_0^{(L)}$ to the eigenvalue $E_n(H_0^{(L)})$, as can be checked by the min-max-principle. Thus $|\varphi_0|^2$ is strictly positive outside a set of Lebesgue measure zero. Therefore $\int |\varphi_0(x)|^2 V(x) > 0$. But this contradicts $\langle \varphi_0, H_0^{(L)} \varphi_0 \rangle = \langle \varphi_0, H_1^{(L)} \varphi_0 \rangle$, thus proving the proposition. \square

REMARK. The poof of Proposition 2 applies to more general situations than described above.

4. **The proof of (ii).** Again it suffices to prove the statement for $d = 2$. Set $A = \{n \in \mathbf{N} \mid n = a^2 + b^2 \text{ for } a, b \in \mathbf{Z}_{++}\}$ and $\tilde{A} = \{p/q \mid p, q \in A\}$.

PROPOSITION 3. If $\lambda \in \tilde{A}$ then $\sup_{L \in \mathbf{N}} \varphi_L(\lambda) = \infty$.

PROOF. Suppose $\lambda = p/q$ with $p, q \in A$, then

$$\lambda = \frac{1}{5^{2k} q^2} 5^{2k} qp = \frac{1}{L^2} \alpha$$

with $L = 5^k q$ and $\alpha = 5^{2k} qp$. Using Gauss' theorem we easily see that $\alpha \in A$, so that λ is an eigenvalue of $H_0^{(L)}$. Again by Gauss' theorem the multiplicity of this eigenvalue increases to infinity as $L \rightarrow \infty$. Thus, by Proposition 2, we have $\lim_{k \rightarrow \infty} \varphi_{5^k q}(\lambda) = \infty$. \square

It remains to show

PROPOSITION 4. \tilde{A} is dense in $[0, \infty)$.

To see this we will make use of the following lemma:

LEMMA. For any $p \in \mathbf{Z}_{++}$ there exists an $m \in A$ such that $p \leq m \leq p + 2(p/2)^{1/2} + 1$.

PROOF. If $n = a^2 + b^2$ with $a \leq b$ then $a \leq (n/2)^{1/2}$; define $n' = (a+1)^2 + b^2 = n + 2a + 1$. Then

$$n' - n = 2a + 1 \leq 2(n/2)^{1/2} + 1.$$

Now given $p > 2$ set $n_p = \max\{m \in A \mid m \leq p\}$ then by definition $n'_p > p$. Moreover

$$n' \leq n + 2(n/2)^{1/2} + 1 \leq p + 2(p/2)^{1/2} + 1,$$

which proves the lemma. \square

We are prepared to prove Proposition 4:

PROOF. First we prove that any integer $r > 0$ can be approximated by certain $r_n \in \tilde{A}$. Indeed, the Lemma tells us that any of the intervals $I_n = [2^n r, 2^n r + 2(2^{n-1}r)^{1/2} + 1]$ has nonempty intersection with A . Denote the smallest element of A within I_n by m_n and set $r_n = m_n/2^n$. Then $r_n \in \tilde{A}$ and

$$|r_n - r| = m_n/2^n - r = \frac{1}{2^n}(m_n - 2^n r) \leq \frac{1}{2^n}(2(2^{n-1}r)^{1/2} + 1) \rightarrow 0.$$

Now let $r = p/q$ be an arbitrary positive rational number. By multiplying, if necessary, p and q by the same number, we may assume without loss of generality that $q \in A$. By the above considerations we have $p_n \in \tilde{A}$ approaching p as $n \rightarrow \infty$. Set $r_n = p_n/q$; then r_n belongs to \tilde{A} . Moreover $r_n = p_n/q \rightarrow p/q = r$. \square

5. A consequence. Let us denote by $\Delta_{C_L - C_1}^{D,N}$ the Laplacian on $C_L - C_1$ with Dirichlet conditions at the outer boundary, i.e. at ∂C_L , and with Neumann condition at the inner boundary, i.e. at ∂C_1 . By $\Delta_{C_L - C_1}^{D,D}$ we denote the Laplacian on the same region with Dirichlet conditions on the whole boundary. As above one might argue that

$$\tilde{\varphi}_L(\lambda) = N(-\Delta_{C_L - C_1}^{D,N}, \lambda) - N(-\Delta_{C_L - C_1}^{D,D}, \lambda)$$

should be bounded, since the surface where the boundary condition is changed remains the same for all L . However, the same assertions as in Theorem 1 are valid for φ_L replaced by $\tilde{\varphi}_L$. The reason is a simple Dirichlet-Neumann bracketing argument (see Reed-Simon [2]):

$$\begin{aligned} \varphi_L(\lambda) &= N(-\Delta_{C_L}^D, \lambda) - N(-\Delta_{C_L}^D + V, \lambda) \\ &\leq N(-\Delta_{C_L - C_1}^{D,N} + -\Delta_{C_1}^N, \lambda) - N(-\Delta_{C_L - C_1}^{D,D} + (-\Delta_{C_1}^D + V), \lambda) \\ &= N(-\Delta_{C_L - C_1}^{D,N}, \lambda) - N(-\Delta_{C_L - C_1}^{D,D}, \lambda) + N(-\Delta_{C_1}^N, \lambda) - N(-\Delta_{C_1}^D + V, \lambda) \\ &\leq \tilde{\varphi}_L(\lambda) + (N(-\Delta_{C_1}^N, \lambda) - N(-\Delta_{C_1}^D, \lambda)). \end{aligned}$$

Since the difference above is independent of L the result follows.

REFERENCES

1. F. Fricker, *Einführung in die Gitterpunktlehre*, Birkhäuser, 1982.
2. M. Reed and B. Simon, *Methods of modern mathematical physics*, Vol. IV, Academic Press, 1978.