NORMAL FILTERS GENERATED BY A FAMILY OF SETS

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ABSTRACT. We study normal filters on the set spaces $\lambda$, $\mathcal{P}_\lambda(\lambda)$, $|\lambda|^\kappa$, and $(\lambda)^\kappa$. We characterize the least normal $\gamma$-complete filter containing a given $\gamma$-complete filter for $\gamma \geq \omega_1$. If $\mathcal{F}$ is a $\omega_1$-complete filter on any of the set spaces mentioned, the least $\omega_1$-complete normal filter containing it is the filter generated by the sets $\{x \in E \mid a_1, \ldots, a_n \in x \rightarrow x \in f(a_1, \ldots, a_n)\}$ where $f: \lambda^{<\omega} \rightarrow \mathcal{F}$.

1. Introduction. Let $\kappa < \lambda$ be uncountable regular cardinals. The sets $\lambda$, $\mathcal{P}_\lambda(\lambda) = \{x \subseteq \lambda \mid |x| < \kappa\}$, $|\lambda|^\kappa = \{x \subseteq \lambda \mid |x| = \kappa\}$, and $(\lambda)^\kappa = \{x \in E \mid \bar{x} = \kappa\}$ are intimately related to certain large cardinal properties. Nevertheless, even without making any large cardinal hypotheses they offer a very rich combinatorial structure interesting by itself.

In this paper we study several normal filters on these spaces built over $\omega_1$-complete families of subsets. Throughout the paper we will use $E$ to denote any of the set spaces mentioned above. A family $S \subseteq \mathcal{P}(E)$ is $\gamma$-complete if it is closed under intersections of size $< \gamma$. Let $\{X_\xi \mid \xi < \lambda\}$ be a sequence of subsets of $E$. The diagonal intersection of the sequence is defined, as usual, by

$$\Delta_{\xi<\lambda} X_\xi = \{x \in E \mid (\forall \xi \in x)(x \in X_\xi)\}.$$

The filter generated by a family $L$ of subsets of $E$ closed under intersections is $[L] = \{A \subseteq E \mid A \supseteq B$ for some $B \in L\}$. The filter $\omega_1$-generated by $L$ is the collection of all those subsets of $L$ which contain a countable intersection of elements of $L$.

A filter $\mathcal{F}$ on $E$ is nontrivial if $\varnothing \not\in \mathcal{F}$, and $\mathcal{F}$ is normal if it is closed under diagonal intersections, in other words, if $\{X_\xi \mid \xi < \lambda\} \subseteq \mathcal{F}$ implies $\Delta_{\xi<\lambda} X_\xi \in \mathcal{F}$.

A filter is subnormal if it is contained in a nontrivial normal filter.

Given a filter $\mathcal{F}$ on $E$, a subset $S \subseteq E$ is said to have positive $\mathcal{F}$-measure if $S \cap A \neq \varnothing$ for all $A \in \mathcal{F}$. A set $S \subseteq E$ has $\mathcal{F}$-measure 0 if it is not of positive $\mathcal{F}$-measure.

A function $f: E \rightarrow \lambda$ is called regressive on $X \subseteq E$ if for all $x \in X$ ($x \neq \varnothing$) $f(x) \in x$. A function $f: E \rightarrow A$ taking values in any set $A$ is $\mathcal{F}$-small if for every $a \in A$, $f^{-1}(a)$ has $\mathcal{F}$-measure 0.

The following statement is a reformulation of Fodor's theorem [F] and is due to Jech [Je] for the closed unbounded filter on $\mathcal{P}_\lambda(\lambda)$ and to Solovay, Reinhardt, and Kanamori [SRK] for the case of ultrafilters on $(\lambda)^\kappa$.

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THEOREM 1.1. Let $\mathcal{F}$ be a filter on $E$. Then $\mathcal{F}$ is normal if and only if for every function $f : E \to \lambda$ regressive on a positive $\mathcal{F}$-measure set $S \subseteq E$ there is $\alpha < \lambda$ and $S' \subseteq S$ also of positive $\mathcal{F}$-measure such that $f(x) = \alpha$ for all $x \in S'$.

PROOF. In one direction just follow [Je, 3.2(d)]. The converse follows from the definition of diagonal intersection. $\square$

In the next section we prove some facts concerning iterations of the diagonal intersection operation which will be used to characterize the least normal $\omega_1$-complete extension of an $\omega_1$-complete filter on a set space $E$.

In §3 we give this characterization and generalize to our spaces a result of Baumgartner, Taylor, and Wagon concerning subnormal filters. The last section contains some examples, including some normal filters properly contained in the closed unbounded filter on $\mathcal{P}_\kappa(\lambda)$.

2. Iterating diagonal intersections. Let $L$ be any family of subsets of $E$. We will denote by $\Delta L$ the set $\{\Delta_{\xi<\lambda} X_\xi \mid X_\xi \in L \text{ for all } \xi < \lambda\}$. Inductively we define the iterations of the diagonal intersection operation as follows

$$\Delta^{\alpha+1} L = \Delta(\Delta^\alpha L),$$

and $\Delta^\theta L = \bigcup_{\beta < \theta} \Delta^\beta L$, for $\theta$ a limit ordinal (here, $\Delta^0 L = L$). It will be useful to introduce some notation. If $f : \lambda^n \to L$ then

$$\Delta f = \{x \in E \mid \alpha_1, \ldots, \alpha_n \in x \to x \in f(\alpha_1, \ldots, \alpha_n)\}.$$  

Notice that for every $n$, $\Delta^n L$ is just the set $\{\Delta f \mid f : \lambda^n \to L\}$.

Analogously, if $f : \lambda^{<\omega} \to L$, define

$$\Delta f = \{x \in E \mid s \in x^{<\omega} \to x \in f(s)\}$$

and

$$\Delta^{<\omega} L = \{\Delta f \mid f : \lambda^{<\omega} \to L\}.$$  

It is easy to verify that if $L$ is a $\gamma$-complete family of subsets of $E$ then $\Delta L$ is also $\gamma$-complete. The same holds for $\Delta^{<\omega} L$ but not necessarily for $\Delta^\omega L$.

THEOREM 2.1. If $L$ is a $\gamma$-complete family of subsets of $E$, and $\gamma \geq \omega_1$, then $\{\bigcap_{n \in \omega} X_n \mid X_n \in \Delta^n L\}$ generates the least $\omega_1$-complete filter extending $\Delta^\omega L$; moreover, it is $\gamma$-complete.

PROOF. Let $\mathcal{G}$ be the filter generated by the collection $\{\bigcap_{n \in \omega} X_n \mid X_n \in \Delta^n L\}$. To show that $\mathcal{G}$ is $\gamma$-complete, let $\delta < \gamma$ and let $\{Y_\xi \mid \xi < \delta\}$ be a collection of elements of $\mathcal{G}$. For every $\xi < \delta$, $Y_\xi \supseteq \bigcap_{n \in \omega} X_\xi^n$ for some sequence $\{X_\xi^n \mid n \in \omega\}$ such that $X_\xi^n \in \Delta^n L$ for every $n \in \omega$. Put $Z_n = \bigcap_{\xi < \delta} X_\xi^n$. Since $\Delta^n L$ is $\gamma$-complete, $Y_n \in \Delta^n L$. Clearly,

$$\bigcap_{\xi < \delta} \bigcap_{n \in \omega} X_\xi^n = \bigcap_{n \in \omega} Z_n \in \mathcal{G}.$$  

If $\mathcal{F}$ is any $\omega_1$-complete filter extending $\bigcup_{n \in \omega} \Delta^n L$, then $\bigcap_{n \in \omega} X_n \in \mathcal{F}$ for each family $\{X_n \mid n \in \omega\}$ such that $X_n \in \Delta^n L$ for every $n \in \omega$. Therefore the filter generated by $\{\bigcap_{n \in \omega} X_n \mid X_n \in \Delta^n L\}$ is contained in $\mathcal{F}$. $\square$
3. Normal extensions. A very natural problem is to characterize the least normal filter with a given degree of completeness and containing a given family of subsets of \( E \).

Clearly, \( \Delta^{\lambda^+} \) generates the least normal filter containing \( L \), but our next result shows that if \( L \) is an \( \omega_1 \)-complete family of subsets of \( E \), we only need to look at the first \( \omega \) iterations of the diagonal intersection operation to obtain an \( \omega_1 \)-complete normal filter extending \( L \).

**Theorem 3.1.** For every \( L \subseteq \mathcal{P}(E) \), \( \Delta^{<\omega} L \) generates the least normal \( \omega_1 \)-complete filter on \( E \) extending \( L \).

**Proof.** We show that the family \( \Delta^{<\omega} L \) is itself \( \omega_1 \)-complete. Let \( \{X_i \mid i \in \omega\} \subseteq \Delta^{<\omega} L \). For each \( i \in \omega \) there is a function \( f_i : \lambda^{<\omega} \rightarrow L \) such that \( \Delta f_i = X_i \).

It is easy to code the functions \( f_i \) into a single function \( f \) such that \( \Delta f \subseteq \Delta f_i \) for each \( i \). This can be done, for example, defining

\[
f(\alpha_1, \ldots, \alpha_n) = f_i(\alpha_1, \ldots, \alpha_n),
\]

where

\[
n = p_i^{n_i} p_{i+1}^{n_{i+1}} \cdots p_{i+j}^{n_{i+j}}.
\]

To prove that \( \Delta^{<\omega} L \) generates a normal filter, let \( \{f_\xi \mid \xi < \lambda\} \) be a family of functions, each \( f_\xi : \lambda^{<\omega} \rightarrow L \). Define \( f : \lambda^{<\omega} \rightarrow L \) by

\[
f(\alpha_1, \ldots, \alpha_n) = f_{\alpha_1}(\alpha_2, \alpha_3, \ldots, \alpha_n) \quad \text{if } n > 1
\]

(and \( f(\alpha) = f_\alpha(\alpha) \) otherwise).

Clearly \( \Delta f \subseteq \Delta \xi(\Delta f_\xi) \).

Suppose \( \mathcal{F} \) is a normal \( \omega_1 \)-complete filter extending \( L \) and strictly contained in \( \{\Delta^{<\omega} L\} \). Then, there is a function \( f : \lambda^{<\omega} \rightarrow L \) such that \( \Delta f \in \{\Delta^{<\omega} L\} - \mathcal{F} \). The set \( E - \Delta f \) is of positive \( \mathcal{F} \)-measure. Consider the function \( h : E - \Delta f \rightarrow \lambda^{<\omega} \) defined by \( h(x) = \text{some } s \in [x]^{<\omega} \) such that \( x \notin f(s) \). By the \( \omega_1 \)-completeness of \( \mathcal{F} \) there is \( n \in \omega \) and \( A \subseteq E - \Delta f \), \( A \) of positive \( \mathcal{F} \)-measure such that \( h \upharpoonright A : A \rightarrow \lambda^n \) and for all \( x \in A \) \( h(x) \in [x]^n \). Applying the normality of \( \mathcal{F} \) \( n \) consecutive times we can find \( s \in \lambda^n \) and \( B \subseteq A \), \( B \) of positive \( \mathcal{F} \)-measure, such that for all \( x \in B \), \( x \notin f(s) \) and thus \( B \cap f(s) = \emptyset \). This contradicts \( f(s) \in L \). \( \square \)

**Corollary 3.2.** If \( L \) is a \( \gamma \)-complete family of subsets of \( E \) and \( \gamma \geq \omega_1 \), then \( \Delta^{\omega} L \) \( \omega_1 \)-generates the least normal \( \gamma \)-complete filter extending \( L \).

**Proof.** Let \( \mathcal{F} \) be the filter \( \omega_1 \)-generated by \( \Delta^{\omega} L \). Theorem 2.1 indicates that \( \mathcal{F} = \{\prod X_n \mid X_n \in \Delta^n L\} \), and so we only need to show that it is normal. By Theorem 3.1, this will be accomplished if we prove that \( \mathcal{F} \) is generated by \( \Delta^{<\omega} L \).

Let \( A \in \mathcal{F} \); there are functions \( f_n : \lambda^n \rightarrow L \), \( n \in \omega \), such that \( A \supseteq \bigcap_{n \in \omega} X_n \) where, for each \( n \in \omega \), \( X_n = \Delta f_n \).

Define \( f : \lambda^{<\omega} \rightarrow L \) by \( f(\alpha_1, \ldots, \alpha_n) = f_n(\alpha_1, \ldots, \alpha_n) \). Then \( A \supseteq \Delta f \) and thus \( A \in \Delta^{<\omega} L \).

Conversely, if \( A \in \Delta^{<\omega} L \), \( A \supseteq \Delta f \) for some \( f : \lambda^{<\omega} \rightarrow L \). For each \( n \in \omega \), put \( f_n = f \upharpoonright \lambda^n \), then \( \Delta f_n = \bigcap_{n \in \omega} \Delta f_n \). Since \( \Delta f_n \in \Delta^n L \) for each \( n \in \omega \), we conclude that \( A \in \mathcal{F} \). \( \square \)
Baumgartner, Taylor, and Wagon proved in [BTW] a result characterizing, in terms of regressive functions, subnormal, nonprincipal, $\lambda$-complete ideals on a regular cardinal $\lambda$. As a consequence of our previous results this characterization extends to all set spaces $E$.

**COROLLARY 3.3.** The following are equivalent for an $\omega_1$-complete filter $\mathcal{F}$ on $E$:

(i) $\mathcal{F}$ is contained in a nontrivial $\omega_1$-complete normal filter.

(ii) If $f: E \to \lambda^{<\omega}$ is regressive (i.e. for all $x \in E$, $f(x)$ is a finite sequence of elements of $x$), then it is constant on a set of positive $\mathcal{F}$-measure.

(iii) If $A \in \mathcal{F}$ then there is no $\mathcal{F}$-small regressive function $f: A \to \lambda^{<\omega}$.

**PROOF.** For the equivalence of (i) and (ii), let $\mathcal{F}$ be contained in a nontrivial $\omega_1$-complete normal filter and let $f: E \to \lambda^{<\omega}$ be regressive. Suppose $f^{-1}(s)$ is of $\mathcal{F}$-measure 0 for each $s \in \lambda^{<\omega}$. Then $C_s = E - f^{-1}(s)$ belongs to $\mathcal{F}$ for each $s \in \lambda^{<\omega}$ and therefore

$$\Delta_n C_s = \{x \in E \mid s \in x^{<\omega} \rightarrow x \in C_s\}$$

is nonempty (since it belongs to the filter $\Delta^{<\omega} \mathcal{F}$ and this is the least $\omega_1$-complete normal filter extending $\mathcal{F}$). For $x \in \Delta_n C_s$, $f(x) \neq s$ for all $s \in x^{<\omega}$, but this is impossible since $f$ is regressive. Therefore, $f$ must be constant on a set of positive $\mathcal{F}$-measure.

Conversely, if there is no nontrivial $\omega_1$-complete normal filter extending $\mathcal{F}$, then $\Delta^{<\omega} \mathcal{F}$ is trivial and thus there is $f: \lambda^{<\omega} \to \mathcal{F}$ such that $\Delta f = \emptyset$. For every $x \in E$ there is $s \in X^{<\omega}$ such that $x \in f(s)$. Define $h: E \to \lambda^{<\omega}$ by picking one such $s$ for each $x \in E$. This function is regressive but not constant on any set of positive $\mathcal{F}$-measure.

The equivalence between (ii) and (iii) is equally easy to verify. □

In some cases it is possible to generate a normal extension of a family of subsets of $E$ by iterating the diagonal intersection operation a finite number of times.

For example, if $\mathcal{F}$ is a $\lambda$-complete filter on $E$, then $\Delta \mathcal{F}$ is normal. (Given any $f: \lambda^2 \to \mathcal{F}$, define $g: \lambda \to \mathcal{F}$ by $g(\alpha) = \bigcap_{\gamma, \beta \leq \alpha} f(\gamma, \beta)$, then $\Delta g \subseteq \Delta f$.)

Also, if we put $C(\phi) = \{x \in E \mid \forall \alpha, \beta \in x \phi(\alpha, \beta) \in x\}$ for any function $\phi: \lambda^2 \to \lambda$ we have

**PROPOSITION 3.4.** Let $\mathcal{F}$ be a filter on $E$. If for some $n \geq 1$ $C(\phi) \subseteq \Delta^n \mathcal{F}$ for some bijection $\phi: \lambda^2 \to \lambda$ then $\Delta^n \mathcal{F}$ is normal.

**PROOF.** It is enough to show that under the hypothesis of the proposition, for every function $f: \lambda^{n+1} \to \mathcal{F}$, there is a function $g: \lambda^n \to \mathcal{F}$ such that $\Delta g \cap C(\phi) \subseteq \Delta f$. Let $f: \lambda^{n+1} \to \mathcal{F}$ and define $g: \lambda^n \to \mathcal{F}$ by

$$g(\alpha_1, \ldots, \alpha_n) = f(\phi^{-1}(\alpha_1), \alpha_2, \ldots, \alpha_n).$$

Let $x \in \Delta g \cap C(\phi)$. If $\alpha_0, \ldots, \alpha_n \in x$ then $\phi(\alpha_0, \alpha_1) \in x$ and thus,

$$x \in g(\phi(\alpha_0, \alpha_1), \alpha_2, \ldots, \alpha_n) = f(\alpha_0, \ldots, \alpha_n).$$

It can be shown that this is not a necessary condition for the normality of $\Delta^n \mathcal{F}$ (see 4.1).
4. Some examples.

4.1. Consider the family of the sets $C_\alpha = \{x \in E \mid \exists \beta \in x \ (\beta > \alpha)\}$. This family generates a $\lambda$-complete filter and therefore, $[\Delta(C_\alpha \mid \alpha < \lambda)]$ is normal.

Note that for the case $E = \mathcal{P}_\kappa(\lambda)$, the normal filter generated by $\Delta(C_\alpha \mid \alpha < \lambda)$ is properly contained in the closed unbounded filter. This is so because each diagonal intersection of sets of the form $C_\alpha$ is closed under arbitrary unions and unbounded. This example confirms the remark after Proposition 3.4, since no diagonal intersection $\Delta \mathcal{C}_\alpha$ is contained in $C(j)$ for a bijection $j: \lambda \times \lambda \to \lambda$ (see [C, 2.6]).

4.2. Consider the sets $C_p = \{x \in E \mid p \subseteq x\}$, for $p \in \mathcal{P}_\kappa(\lambda)$. The filter generated by these sets is $\kappa$-complete (but not $\kappa^+$-complete). For the case $E = \lambda$ these are just the sets considered in 4.1.

For $E = \mathcal{P}_\kappa(\lambda)$, $C_p = \hat{p} = \{q \in \mathcal{P}_\kappa(\lambda) \mid p \subseteq q\}$, and by results of [C],

$$\left[\{\hat{p} \mid p \in \mathcal{P}_\kappa(\lambda)\}\right] \subset \left[\Delta\{\hat{p} \mid p \in \mathcal{P}_\kappa(\lambda)\}\right] \subset \left[\Delta^2\{\hat{p} \mid p \in \mathcal{P}_\kappa(\lambda)\}\right] = \text{closed unbounded filter}.$$  

The case $E = [\lambda]^\kappa$ was studied in [DP.M1] and [DP.M2]. In this case we have $C_p = \hat{p} = \{Q \in [\lambda]^\kappa \mid p \subseteq Q\}$ and also, by Proposition 3.4, $\Delta^2\{\hat{p} \mid p \in \mathcal{P}_\kappa(\lambda)\}$ is normal. The situation for $E = (\lambda)^\kappa$ is completely different, ZFC $\not\vdash$ “there is a nontrivial $\kappa$-complete normal filter on $(\lambda)^\kappa$ containing the sets $C_p = \{Q \in (\lambda)^\kappa \mid p \subseteq Q\}$. [DP.M2].

4.3. Consider the family of sets

$$O(\gamma) = \{x \in E \mid \text{order type of } x > \gamma\}$$

for $\gamma < \lambda$.

This family is interesting for $E = \mathcal{P}_\kappa(\lambda)$ and $\gamma < \kappa$. A subset $A \subseteq \mathcal{P}_\kappa(\lambda)$ is unbounded by final segments if for all $p \in \mathcal{P}_\kappa(\lambda)$ there is $\alpha < \lambda$ such that $p \cup (\alpha - \cup p) \in A$. The filter generated by the closed sets which are unbounded by final segments is properly contained in the closed unbounded filter (see [DP.M2]).

**Theorem 4.3.1.** Let $\mathcal{F}_0$ be the filter on $\mathcal{P}_\kappa(\lambda)$ generated by $\{O(\gamma) \mid \gamma < \kappa\}$,

$$\mathcal{F}_0 \subset \mathcal{F}_1,$$

where $\mathcal{F}_1$ is the filter generated by the closed sets which are unbounded by final segments.

**Proof.** For the first proper inclusion consider the set

$$A = \{p \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in p \cap \kappa \rightarrow \alpha + 1 < p\}.$$  

Clearly, $A \not\subset \{O(\gamma) \mid \gamma < \kappa\}$, and $A = \Delta_{\xi<\kappa} A_\xi$ where $A_\xi = O(\xi + 1)$ if $\xi < \kappa$ and $A_\xi = O(0)$ if $\xi \geq \kappa$.

For the second proper inclusion, consider the set $B = \{p \in \mathcal{P}_\kappa(\lambda) \mid \text{for some } \alpha < \lambda, I_\alpha \cap p \text{ contains at least two elements}\}$, where $I_\alpha = [\kappa \cdot \alpha, \kappa \cdot (\alpha + 1))$. This set is closed and unbounded by final segments, but it does not contain any set in $\Delta^{\omega}O(\gamma) \mid \gamma < \kappa$. □

We do not know if there is $n \geq 1$ such that

$$\Delta^n O(\gamma) \mid \gamma < \kappa = \Delta^\omega O(\gamma) \mid \gamma < \kappa.$$
Further study of these and other filters on $\mathcal{P}_\kappa(\lambda)$ will be undertaken in a forthcoming paper. There it will be shown that, in general, there is no finite bound to the number of iterations of the diagonal intersection operation needed to obtain a normal filter from a filter on $E$.

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