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NORMAL FILTERS GENERATED BY A FAMILY OF SETS

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ABSTRACT. We study normal filters on the set spaces λ, $\mathcal{P}_\kappa(\lambda)$, $[\lambda]^\kappa$, and $(\lambda)^\kappa$. We characterize the least *normal* γ-complete filter containing a given γ-complete filter for $\gamma \geq \omega_1$. If $\mathcal{F}$ is a ω_1-complete filter on any of the set spaces mentioned, the least ω_1-complete normal filter containing it is the filter generated by the sets $\{x \in E \mid \alpha_1, \ldots, \alpha_n \in x \to x \in f(\alpha_1, \ldots, \alpha_n)\}$ where $f\colon \lambda^{<\omega} \to \mathcal{F}$.

1. Introduction. Let $\kappa < \lambda$ be uncountable regular cardinals. The sets λ, $\mathcal{P}_\kappa(\lambda) = \{x \subseteq \lambda \mid |x| < \kappa\}$, $[\lambda]^\kappa = \{x \subseteq \lambda \mid |x| = \kappa\}$, and $(\lambda)^\kappa = \{x \in E \mid \overline{x} = \kappa\}$ are intimately related to certain large cardinal properties. Nevertheless, even without making any large cardinal hypotheses they offer a very rich combinatorial structure interesting by itself.

In this paper we study several normal filters on these spaces built over ω_1-complete families of subsets. Throughout the paper we will use E to denote any of the set spaces mentioned above. A family $S \subseteq \mathcal{P}(E)$ is γ-complete if it is closed under intersections of size $< \gamma$. Let $\{X_\xi \mid \xi < \lambda\}$ be a sequence of subsets of E. The diagonal intersection of the sequence is defined, as usual, by

$$\Delta_{\xi<\lambda} X_\xi = \{x \in E \mid (\forall \xi \in x)(x \in X_\xi)\}.$$

The filter generated by a family L of subsets of E closed under intersections is $[L] = \{A \subseteq E \mid A \supseteq B \text{ for some } B \in L\}$. The filter ω_1-generated by L is the collection of all those subsets of L which contain a countable intersection of elements of L.

A filter $\mathcal{F}$ on E is nontrivial if $\phi \notin \mathcal{F}$, and $\mathcal{F}$ is normal if it is closed under diagonal intersections, in other words, if $\{X_\xi \mid \xi < \lambda\} \subseteq \mathcal{F}$ implies $\Delta_{\xi<\lambda} X_\xi \in \mathcal{F}$.

A filter is subnormal if it is contained in a nontrivial normal filter.

Given a filter $\mathcal{F}$ on E, a subset $S \subseteq E$ is said to have positive $\mathcal{F}$-measure if $S \cap A \neq \emptyset$ for all $A \in \mathcal{F}$. A set $S \subseteq E$ has $\mathcal{F}$-measure 0 if it is not of positive $\mathcal{F}$-measure.

A function $f\colon E \to \lambda$ is called regressive on $X \subseteq E$ if for all $x \in X$ $(x \neq \emptyset)$ $f(x) \in x$. A function $f\colon E \to A$ taking values in any set A is $\mathcal{F}$-small if for every $a \in A$, $f^{-1}(a)$ has $\mathcal{F}$-measure 0.

The following statement is a reformulation of Fodor's theorem [**F**] and is due to Jech [**Je**] for the closed unbounded filter on $\mathcal{P}_\kappa(\lambda)$ and to Solovay, Reinhardt, and Kanamori [**SRK**] for the case of ultrafilters on $(\lambda)^\kappa$.

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THEOREM 1.1. *Let $\mathcal{F}$ be a filter on E. Then $\mathcal{F}$ is normal if and only if for every function $f\colon E \to \lambda$ regressive on a positive $\mathcal{F}$-measure set $S \subseteq E$ there is $\alpha < \lambda$ and $S' \subseteq S$ also of positive $\mathcal{F}$-measure such that $f(x) = \alpha$ for all $x \in S'$.*

PROOF. In one direction just follow [**Je**, 3.2(d)]. The converse follows from the definition of diagonal intersection. □

In the next section we prove some facts concerning iterations of the diagonal intersection operation which will be used to characterize the least normal ω_1-complete extension of an ω_1-complete filter on a set space E.

In §3 we give this characterization and generalize to our spaces a result of Baumgartner, Taylor, and Wagon concerning subnormal filters. The last section contains some examples, including some normal filters properly contained in the closed unbounded filter on $\mathcal{P}_\kappa(\lambda)$.

2. Iterating diagonal intersections. Let L be any family of subsets of E. We will denote by ΔL the set $\{\Delta_{\xi<\lambda} X_\xi \mid X_\xi \in L \text{ for all } \xi < \lambda\}$. Inductively we define the iterations of the diagonal intersection operation as follows

$$\Delta^{\alpha+1} L = \Delta(\Delta^\alpha L),$$

and $\Delta^\theta L = \bigcup_{\beta<\theta} \Delta^\beta L$, for θ a limit ordinal (here, $\Delta^0 L = L$). It will be useful to introduce some notation. If $f\colon \lambda^n \to L$ then

$$\Delta f = \{x \in E \mid \alpha_1, \ldots, \alpha_n \in x \to x \in f(\alpha_1, \ldots, \alpha_n)\}.$$

Notice that for every n, $\Delta^n L$ is just the set $\{\Delta f \mid f\colon \lambda^n \to L\}$.

Analogously, if $f\colon \lambda^{<\omega} \to L$, define

$$\Delta f = \{x \in E \mid s \in x^{<\omega} \to x \in f(s)\}$$

and

$$\Delta^{<\omega} L = \{\Delta f \mid f\colon \lambda^{<\omega} \to L\}.$$

It is easy to verify that if L is a γ-complete family of subsets of E then ΔL is also γ-complete. The same holds for $\Delta^{<\omega} L$ but not necessarily for $\Delta^\omega L$.

THEOREM 2.1. *If L is a γ-complete family of subsets of E, and $\gamma \geq \omega_1$, then $\{\bigcap_{n\in\omega} X_n \mid X_n \in \Delta^n L\}$ generates the least ω_1-complete filter extending $\Delta^\omega L$; moreover, it is γ-complete.*

PROOF. Let $\mathcal{G}$ be the filter generated by the collection $\{\bigcap_{n\in\omega} X_n \mid X_n \in \Delta^n L\}$. To show that $\mathcal{G}$ is γ-complete, let $\delta < \gamma$ and let $\{Y_\xi \mid \xi < \delta\}$ be a collection of elements of $\mathcal{G}$. For every $\xi < \delta$, $Y_\xi \supseteq \bigcap_{n\in\omega} X_n^\xi$ for some sequence $\{X_n^\xi \mid n \in \omega\}$ such that $X_n^\xi \in \Delta^n L$ for every $n \in \omega$. Put $Z_n = \bigcap_{\xi<\delta} X_n^\xi$. Since $\Delta^n L$ is γ-complete, $Y_n \in \Delta^n L$. Clearly,

$$\bigcap_{\xi<\delta} Y_\xi \supseteq \bigcap_{n\in\omega} \bigcap_{\xi<\delta} X_n^\xi = \bigcap_{n\in\omega} Z_n \in \mathcal{G}.$$

If $\mathcal{F}$ is any ω_1-complete filter extending $\bigcup_{n\in\omega} \Delta^n L$, then $\bigcap_{n\in\omega} X_n \in \mathcal{F}$ for each family $\{X_n \mid n \in \omega\}$ such that $X_n \in \Delta^n L$ for every $n \in \omega$. Therefore the filter generated by $\{\bigcap_{n\in\omega} X_n \mid X_n \in \Delta^n L\}$ is contained in $\mathcal{F}$. □

3. Normal extensions. A very natural problem is to characterize the least normal filter with a given degree of completeness and containing a given family of subsets of E.

Clearly, $\Delta^{\lambda^+} L$ generates the least normal filter containing L, but our next result shows that if L is an ω_1-complete family of subsets of E, we only need to look at the first ω iterations of the diagonal intersection operation to obtain an ω_1-complete normal filter extending L.

THEOREM 3.1. *For every $L \subseteq \mathcal{P}(E)$, $\Delta^{<\omega} L$ generates the least normal ω_1-complete filter on E extending L.*

PROOF. We show that the family $\Delta^{<\omega} L$ is itself ω_1-complete. Let $\{X_i \mid i \in \omega\} \subseteq \Delta^{<\omega} L$. For each $i \in \omega$ there is a function $f_i \colon \lambda^{<\omega} \to L$ such that $\Delta f_i = X_i$. It is easy to code the functions f_i into a single function f such that $\Delta f \subseteq \Delta f_i$ for each i. This can be done, for example, defining

$$f(\alpha_1, \ldots, \alpha_n) = f_i(\alpha_1, \ldots, \alpha_{n_i}),$$

where

$$n = p_i^{n_i} p_{i+1}^{n_{i+1}} \cdots p_{i+j}^{n_{i+j}}.$$

To prove that $\Delta^{<\omega} L$ generates a normal filter, let $\{f_\xi \mid \xi < \lambda\}$ be a family of functions, each $f_\xi \colon \lambda^{<\omega} \to L$. Define $f \colon \lambda^{<\omega} \to L$ by

$$f(\alpha_1, \ldots, \alpha_n) = f_{\alpha_1}(\alpha_2, \alpha_3, \ldots, \alpha_n) \quad \text{if } n > 1$$

$$(\text{and } f(\alpha) = f_\alpha(\alpha) \text{ otherwise}).$$

Clearly $\Delta f \subseteq \Delta_\xi(\Delta f_\xi)$.

Suppose $\mathcal{F}$ is a normal ω_1-complete filter extending L and strictly contained in $[\Delta^{<\omega} L]$. Then, there is a function $f \colon \lambda^{<\omega} \to L$ such that $\Delta f \in [\Delta^{<\omega} L] - \mathcal{F}$. The set $E - \Delta f$ is of positive $\mathcal{F}$-measure. Consider the function $h \colon E - \Delta f \to \lambda^{<\omega}$ defined by $h(x) =$some $s \in [x]^{<\omega}$ such that $x \notin f(s)$. By the ω_1-completeness of $\mathcal{F}$ there is $n \in \omega$ and $A \subseteq E - \Delta f$, A of positive $\mathcal{F}$-measure such that $h \restriction A \colon A \to \lambda^n$ and for all $x \in A$ $h(x) \in [x]^n$. Applying the normality of $\mathcal{F}$ n consecutive times we can find $s \in \lambda^n$ and $B \subseteq A$, B of positive $\mathcal{F}$-measure, such that for all $x \in B$, $x \notin f(s)$ and thus $B \cap f(s) = \emptyset$. This contradicts $f(s) \in L$. $\square$

COROLLARY 3.2. *If L is a γ-complete family of subsets of E and $\gamma \geq \omega_1$, then $\Delta^\omega L$ ω_1-generates the least normal γ-complete filter extending L.*

PROOF. Let $\mathcal{F}$ be the filter ω_1-generated by $\Delta^\omega L$. Theorem 2.1 indicates that $\mathcal{F} = [\{\bigcap X_n \mid X_n \in \Delta^n L\}]$, and so we only need to show that it is normal. By Theorem 3.1, this will be accomplished if we prove that $\mathcal{F}$ is generated by $\Delta^{<\omega} L$.

Let $A \in \mathcal{F}$; there are functions $f_n \colon \lambda^n \to L$, $n \in \omega$, such that $A \supseteq \bigcap_{n \in \omega} X_n$ where, for each $n \in \omega$, $X_n = \Delta f_n$.

Define $f \colon \lambda^{<\omega} \to L$ by $f(\alpha_1, \ldots, \alpha_n) = f_n(\alpha_1, \ldots, \alpha_n)$. Then $A \supseteq \Delta f$ and thus $A \in \Delta^{<\omega} L$.

Conversely, if $A \in \Delta^{<\omega} L$, $A \supseteq \Delta f$ for some $f \colon \lambda^{<\omega} \to L$. For each $n \in \omega$, put $f_n = f \mid \lambda^n$, then $\Delta f = \bigcap_{n \in \omega} \Delta f_n$. Since $\Delta f_n \in \Delta^n L$ for each $n \in \omega$, we conclude that $A \in \mathcal{F}$. $\square$

Baumgartner, Taylor, and Wagon proved in [**BTW**] a result characterizing, in terms of regressive functions, subnormal, nonprincipal, λ-complete ideals on a regular cardinal λ. As a consequence of our previous results this characterization extends to all set spaces E.

COROLLARY 3.3. *The following are equivalent for an ω_1-complete filter $\mathcal{F}$ on E:*

(i) *$\mathcal{F}$ is contained in a nontrivial ω_1-complete normal filter.*

(ii) *If $f\colon E \to \lambda^{<\omega}$ is regressive (i.e. for all $x \in E$, $f(x)$ is a finite sequence of elements of x), then it is constant on a set of positive $\mathcal{F}$-measure.*

(iii) *If $A \in \mathcal{F}$ then there is no $\mathcal{F}$-small regressive function $f\colon A \to \lambda^{<\omega}$.*

PROOF. For the equivalence of (i) and (ii), let $\mathcal{F}$ be contained in a nontrivial ω_1-complete normal filter and let $f\colon E \to \lambda^{<\omega}$ be regressive. Suppose $f^{-1}(s)$ is of $\mathcal{F}$-measure 0 for each $s \in \lambda^{<\omega}$. Then $C_s = E - f^{-1}(s)$ belongs to $\mathcal{F}$ for each $s \in \lambda^{<\omega}$ and therefore

$$\Delta_s C_s = \{x \in E \mid s \in x^{<\omega} \to x \in C_s\}$$

is nonempty (since it belongs to the filter $\Delta^{<\omega}\mathcal{F}$ and this is the least ω_1-complete normal filter extending $\mathcal{F}$). For $x \in \Delta_s C_s$, $f(x) \neq s$ for all $s \in x^{<\omega}$, but this is impossible since f is regressive. Therefore, f must be constant on a set of positive $\mathcal{F}$-measure.

Conversely, if there is no nontrivial ω_1-complete normal filter extending $\mathcal{F}$, then $\Delta^{<\omega}\mathcal{F}$ is trivial and thus there is $f\colon \lambda^{<\omega} \to \mathcal{F}$ such that $\Delta f = \varnothing$. For every $x \in E$ there is $s \in X^{<\omega}$ such that $x \in f(s)$. Define $h\colon E \to \lambda^{<\omega}$ by picking one such s for each $x \in E$. This function is regressive but not constant on any set of positive $\mathcal{F}$-measure.

The equivalence between (ii) and (iii) is equally easy to verify. □

In some cases it is possible to generate a normal extension of a family of subsets of E by iterating the diagonal intersection operation a finite number of times.

For example, if $\mathcal{F}$ is a λ-complete filter on E, then $\Delta\mathcal{F}$ is normal. (Given any $f\colon \lambda^2 \to \mathcal{F}$, define $g\colon \lambda \to \mathcal{F}$ by $g(\alpha) = \bigcap_{\gamma,\beta\leq\alpha} f(\gamma,\beta)$, then $\Delta g \subseteq \Delta f$).

Also, if we put $C(\phi) = \{x \in E \mid \forall \alpha,\beta \in x\ \phi(\alpha,\beta) \in x\}$ for any function $\phi\colon \lambda^2 \to \lambda$ we have

PROPOSITION 3.4. *Let $\mathcal{F}$ be a filter on E. If for some $n \geq 1$ $C(\phi) \in \Delta^n\mathcal{F}$ for some bijection $\phi\colon \lambda^2 \to \lambda$ then $\Delta^n\mathcal{F}$ is normal.*

PROOF. It is enough to show that under the hypothesis of the proposition, for every function $f\colon \lambda^{n+1} \to \mathcal{F}$, there is a function $g\colon \lambda^n \to \mathcal{F}$ such that $\Delta g \cap C(\phi) \subseteq \Delta f$. Let $f\colon \lambda^{n+1} \to \mathcal{F}$ and define $g\colon \lambda^n \to \mathcal{F}$ by

$$g(\alpha_1,\ldots,\alpha_n) = f(\phi^{-1}(\alpha_1),\alpha_2,\ldots,\alpha_n).$$

Let $x \in \Delta g \cap C(\phi)$. If $\alpha_0,\ldots,\alpha_n \in x$ then $\phi(\alpha_0,\alpha_1) \in x$ and thus,

$$x \in g(\phi(\alpha_0,\alpha_1),\alpha_2,\ldots,\alpha_n) = f(\alpha_0,\ldots,\alpha_n). \quad \square$$

It can be shown that this is not a necessary condition for the normality of $\Delta^n\mathcal{F}$ (see 4.1).

4. Some examples.

4.1. Consider the family of the sets $C_\alpha = \{x \in E \mid \exists \beta \in x\ (\beta > \alpha)\}$. This family generates a λ-complete filter and therefore, $[\Delta\{C_\alpha \mid \alpha < \lambda\}]$ is normal.

Note that for the case $E = \mathcal{P}_\kappa(\lambda)$, the normal filter generated by $\Delta\{C_\alpha \mid \alpha < \lambda\}$ is properly contained in the closed unbounded filter. This is so because each diagonal intersection of sets of the form C_α is closed under arbitrary unions and unbounded. This example confirms the remark after Proposition 3.4, since no diagonal intersection $\Delta_\xi C_{\alpha_\xi}$ is contained in $C(j)$ for a bijection $j\colon \lambda \times \lambda \to \lambda$ (see [**C**, 2.6]).

4.2. Consider the sets $C_p = \{x \in E \mid p \subseteq x\}$, for $p \in \mathcal{P}_\kappa(\lambda)$. The filter generated by these sets is κ-complete (but not κ^+-complete). For the case $E = \lambda$ these are just the sets considered in 4.1.

For $E = \mathcal{P}_\kappa(\lambda)$, $C_p = \hat{p} = \{q \in \mathcal{P}_\kappa(\lambda) \mid p \subseteq q\}$, and by results of [**C**],

$$\begin{aligned}[\{\hat{p} \mid p \in \mathcal{P}_\kappa(\lambda)\}] &\subsetneq [\Delta\{\hat{p} \mid p \in \mathcal{P}_\kappa(\lambda)] \\ &\subsetneq [\Delta^2\{\hat{p} \mid p \in \mathcal{P}_\kappa(\lambda)] = \text{closed unbounded filter.}\end{aligned}$$

The case $E = [\lambda]^\kappa$ was studied in [**DP.M1**] and [**DP.M2**]. In this case we have $C_p = \check{p} = \{Q \in [\lambda]^\kappa | p \subseteq Q\}$ and also, by Proposition 3.4, $\Delta^2\{\check{p} \mid p \in \mathcal{P}_\kappa(\lambda)\}$ is normal. The situation for $E = (\lambda)^\kappa$ is completely different, ZFC $\nvdash$ "there is a nontrivial κ-complete normal filter on $(\lambda)^\kappa$ containing the sets $C_p = \{Q \in (\lambda)^\kappa | p \subseteq Q\}$." [**DP.M2**].

4.3. Consider the family of sets

$$O(\gamma) = \{x \in E | \text{order type of } x > \gamma\}$$

for $\gamma < \lambda$.

This family is interesting for $E = \mathcal{P}_\kappa(\lambda)$ and $\gamma < \kappa$. A subset $A \subseteq \mathcal{P}_\kappa(\lambda)$ is unbounded by final segments if for all $p \in \mathcal{P}_\kappa(\lambda)$ there is $\alpha < \lambda$ such that $p \cup (\alpha - \bigcup p) \in A$. The filter generated by the closed sets which are unbounded by final segments is properly contained in the closed unbounded filter (see [**DP.M2**]).

THEOREM 4.3.1. *Let $\mathcal{F}_O$ be the filter on $\mathcal{P}_\kappa(\lambda)$ generated by $\{O(\gamma) \mid \gamma < \kappa\}$,*

$$\mathcal{F}_O \subsetneq [\Delta \mathcal{F}_O] \subseteq [\Delta^{<\omega} \mathcal{F}_0] \subsetneq \mathcal{F}_1,$$

where $\mathcal{F}_1$ is the filter generated by the closed sets which are unbounded by final segments.

PROOF. For the first proper inclusion consider the set

$$A = \{p \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in p \cap \kappa \to \alpha + 1 < \overline{p}\}.$$

Clearly, $A \notin [\{O(\gamma) \mid \gamma < \kappa\}]$, and $A = \Delta_{\xi<\kappa} A_\xi$ where $A_\xi = O(\xi + 1)$ if $\xi < \kappa$ and $A_\xi = O(0)$ if $\xi \geq \kappa$.

For the second proper inclusion, consider the set $B = \{p \in \mathcal{P}_\kappa(\lambda)|$ for some $\alpha < \lambda$, $I_\alpha \cap p$ contains at least two elements$\}$, where $I_\alpha = [\kappa \cdot \alpha, \kappa \cdot (\alpha + 1))$. This set is closed and unbounded by final segments, but it does not contain any set in $\Delta^{<\omega}\{O(\gamma) \mid \gamma < \kappa\}$. $\square$

We do not know if there is $n \geq 1$ such that

$$\Delta^n\{O(\gamma) \mid \gamma < \kappa\} = \Delta^\omega\{O(\gamma) \mid \gamma < \kappa\}.$$

Further study of these and other filters on $\mathcal{P}_\kappa(\lambda)$ will be undertaken in a forthcoming paper. There it will be shown that, in general, there is no finite bound to the number of iterations of the diagonal intersection operation needed to obtain a normal filter from a filter on E.

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