PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 3, November 1987

# CONTINUOUSLY HOMOGENEOUS CONTINUA AND THEIR ARC COMPONENTS

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(Communicated by Doug W. Curtis)

ABSTRACT. Let $X$ be a continuously homogeneous Hausdorff continuum. We prove that if there is a sequence $A_1, A_2, \ldots$ of its arc components with $X = \operatorname{cl} A_1 \cup \operatorname{cl} A_2 \cup \cdots$, and there is an arc component of $X$ with nonempty interior, then $X$ is arcwise connected. As consequences and applications we get: (1) if $X$ is the countable union of arcwise connected continua, then $X$ is arcwise connected; (2) if $X$ is nondegenerate and metric, the number of its arc components is countable and it contains no simple triod, then it is either an arc or a simple closed curve; and, in particular, (3) an arc is the only nondegenerate continuously homogeneous arc-like metric continuum with countably many arc components.

**Introduction.** Recall that a space $X$ is said to be continuously homogeneous if for every two points $x, y \in X$ there is a continuous surjection $f\colon X \to X$ with $f(x) = y$. This notion is due to D. P. Bellamy, and also, in a more general version, to J. J. Charatonik (see **[C]**). The purpose of this paper is to prove that

(1) if a Hausdorff continuum $X$ is continuously homogeneous and there is a sequence $A_1, A_2, \ldots$ of its arc components such that $X = \operatorname{cl} A_1 \cup \operatorname{cl} A_2 \cup \cdots$, and there is an arc component of $X$ with nonempty interior, then $X$ is arcwise connected.

As can be seen, this fact is a strengthening of the result 3 of **[K2**, p. 270]. P. Krupski suggested there that such an improvement (in a somewhat weaker version—see Remark in **[K2**, p. 271]) might be true.

Actually, conclusion (1) is one of the applications of Theorem 1 below. The notion of an arc component is replaced in this theorem by the concept of a $\mathscr{K}$-component defined as follows (compare **[P1]**). Let $X$ be a space and $\mathscr{K}$ be an arbitrary family of subcontinua of $X$ satisfying the two following conditions:

(i) if $K = K_1 \cup K_2$ with $K_1, K_2 \in \mathscr{K}$ and $K_1 \cap K_2 \neq \varnothing$, then $K \in \mathscr{K}$,

(ii) if $K \in \mathscr{K}$ and $f\colon X \to X$ is a continuous surjection, then $f(K) \in \mathscr{K}$.

A set $Y \subset X$ is said to be $\mathscr{K}$-connected if each two of its points lie in a subcontinuum of it belonging to $\mathscr{K}$. The maximal $\mathscr{K}$-connected subsets of the space

Received by the editors January 28, 1986 and, in revised form, August 6, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54F20; Secondary 54F65.
*Key words and phrases*. Continuous homogeneity, covering sequence, Hausdorff continuum, $\mathscr{K}$-component, simple triod.

are called its $\mathscr{K}$-components. It can easily be seen that $\mathscr{K}$-components are arc components when $\mathscr{K}$ is the family of all locally connected metric subcontinua of $X$.

Now we are ready to formulate the theorem.

1. THEOREM. *If a Hausdorff continuum $X$ is continuously homogeneous and there is a sequence $A_1, A_2, \ldots$ of its $\mathscr{K}$-components such that $X = \operatorname{cl} A_1 \cup \operatorname{cl} A_2 \cup \cdots$, and there is a $\mathscr{K}$-component of $X$ with nonempty interior, then $X$ is $\mathscr{K}$-connected.*

To give some possible applications of this theorem, other than one of (1), note that if:

$X$ is planable and $\mathscr{K}$ is the family of all $\delta$-connected subcontinua of $X$ (see [**H1**, **H3** and **H2**]), or

$\mathscr{K}$ is the family of all metric subcontinua of $X$ with index of local disconnectivity less than $\alpha$, for $\alpha < \Omega$ (for the definition see [**P2**, Chapter IV]), or

$\mathscr{K}$ is the family of all weakly chainable metric subcontinua of $X$ (see [**L**]), or

$\mathscr{K}$ is the family of all subcontinua of $X$ that are continuous images of Hausdorff arcs,

then essentially different kinds of $\mathscr{K}$-connectedness are obtained.

In this way, i.e., by considering an arbitrary $\mathscr{K}$-connectedness instead of the arc connectedness only, we may extend a number of results concerning continuous homogeneity (e.g. Propositions 4 and 5 of [**K1**, p. 354], 1 of [**K2**, p. 269], and Theorem 3 of [**CG**, p. 341]).

All spaces considered here are assumed to be Hausdorff. A mapping is a continuous mapping between topological spaces, a surjection is a surjective mapping. An arc is a homeomorphic image of the unit segment $[0, 1]$, a Hausdorff arc is a linearly ordered continuum. Symbol $ab$ denotes an arc with ends $a$ and $b$. The union of three arcs $px$, $py$, and $pz$ is called a simple triod if $px \cap py = px \cap pz = py \cap pz = \{p\}$. A point $e$ is called an end point of a space $X$ if $e \in X$ and for every two arcs $C_1, C_2 \subset X$ both containing $e$ we have either $C_1 \subset C_2$ or $C_2 \subset C_1$. The letters $\omega$ and $\Omega$ denote the first infinite and the first uncountable ordinal, respectively. In this paper, according to [**KM**, p. 235], 0 is considered to be a limit ordinal.

**Covering sequences of compact spaces.** The proof of Theorem 1 makes heavy use of Lemma 3 below. In order to obtain this lemma the notion of a covering sequence of a compact space is employed. This notion is analogous to the concept of a covering sequence of a metric compactum defined in [**P1**]. Let $X$ be a compact space with card $X \geqslant \aleph_0$, and let $\Gamma$ be a limit ordinal of cardinality greater than card $X$. A sequence $\tau = \{X_\alpha\}_{\alpha<\Gamma}$ of compact subsets $X_\alpha$ of $X$ is called a covering sequence of $X$ provided for every $\alpha < \Gamma$ there is a countable ordinal $\beta$ such that $\bigcup\{X_\gamma\colon \alpha \leqslant \gamma \leqslant \alpha + \beta\} = X$. For every covering sequence $\tau = \{X_\alpha\}_{\alpha<\Gamma}$ we inductively define another sequence $\{D_\alpha(\tau)\}_{\alpha<\Gamma}$ of compact subsets of $X$:

$D_0(\tau) = X$,

$D_{\alpha+1}(\tau) = \operatorname{cl}(D_\alpha(\tau) \setminus X_\alpha)$,

$D_\varphi(\tau) = \bigcap\{D_\alpha(\tau)\colon \alpha < \varphi\}$, for each limit ordinal $\varphi > 0$.

Note that the sequence $\{D_\alpha(\tau)\}_{\alpha<\Gamma}$ is decreasing. Observe the following two properties of this sequence.

(2) For every $\alpha < \Gamma$ with $D_\alpha(\tau) \neq \varnothing$ there is $\beta < \Omega$ such that $D_{\alpha+\beta}(\tau) \subsetneqq D_\alpha(\tau)$.

Indeed, let $\beta$ be a number guaranteed by the definition for the number $\alpha$. Then the family $\{D_\alpha(\tau) \cap X_\gamma\colon\ \alpha \leqslant \gamma \leqslant \alpha + \beta\}$ covers the set $D_\alpha(\tau)$, thus, by the Baire theorem, one of its elements $D_\alpha(\tau) \cap X_{\gamma_0}$ has nonempty interior in $D_\alpha(\tau)$. Therefore

$$D_{\alpha+\beta+1}(\tau) \subset D_{\gamma_0+1}(\tau) = \mathrm{cl}\big(D_{\gamma_0}(\tau) \setminus X_{\gamma_0}\big) \subset \mathrm{cl}\big(D_\alpha(\tau) \setminus X_{\gamma_0}\big) \subsetneqq D_\alpha(\tau).$$

(3) There is an ordinal $\beta < \Gamma$ with $D_\beta(\tau) = \varnothing$.

In fact, put $P = \{\alpha < \Gamma\colon\ D_{\alpha+1}(\tau) \subsetneqq D_\alpha(\tau)\}$. Then we have $\operatorname{card} P \leqslant \operatorname{card} X$. By (2) we see that there are only countably many ordinals between any $\alpha \in P$ and its successor in $P$. Therefore cardinality of the ordinal $\sup P$ cannot exceed the cardinal $\aleph_0 \cdot \operatorname{card} P \leqslant \aleph_0 \cdot \operatorname{card} X = \operatorname{card} X$. By the assumption about the cardinality of $\Gamma$ we have $\sup P < \Gamma$ which means that the sets $D_\alpha(\tau)$ for $\alpha > \sup P$ are identical. Thus $D_\alpha(\tau) = \varnothing$ for such $\alpha$ by (2).

The minimum ordinal $\beta$ such that $D_\beta(\tau) = \varnothing$ is denoted by $\lambda(\tau)$.

For further purposes we define a binary operation $*$ between covering sequences of the same space. Let $\tau = \{X_\alpha\}_{\alpha<\Gamma}$, $\psi = \{Y_\alpha\}_{\alpha<\Gamma}$ be covering sequences of $X$. Then we put $\tau*\psi = \{Z_\alpha\}_{\alpha<\Gamma}$, where

$$Z_{2\alpha} = X_\alpha, \qquad Z_{2\alpha+1} = Y_\alpha.$$

Obviously $\tau*\psi$ is also a covering sequence of $X$. Now we prove that

(4) $D_\varphi(\tau*\psi) \subset D_\varphi(\tau) \cap D_\varphi(\psi)$ for every limit ordinal $\varphi < \Gamma$.

PROOF. For $\varphi = 0$ it is a consequence of the definition. Assume (4) is true for a limit ordinal $\varphi$. Thus $D_\varphi(\tau*\psi) \subset D_\varphi(\tau)$, and, using the induction, we prove that for every natural $n$

$$D_{\varphi+2n}(\tau*\psi) \subset D_{\varphi+n}(\tau).$$

Indeed, noting $\varphi + 2n = 2\varphi + 2n = 2(\varphi + n)$, we see that

$$\begin{aligned}
D_{\varphi+2(n+1)}(\tau*\psi) &\subset D_{\varphi+2n+1}(\tau*\psi) = \mathrm{cl}\big(D_{\varphi+2n}(\tau*\psi) \setminus Z_{\varphi+2n}\big)\\
&= \mathrm{cl}\big(D_{\varphi+2n}(\tau*\psi) \setminus Z_{2(\varphi+n)}\big) = \mathrm{cl}\big(D_{\varphi+2n}(\tau*\psi) \setminus X_{\varphi+n}\big)\\
&\subset \mathrm{cl}\big(D_{\varphi+n}(\tau) \setminus X_{\varphi+n}\big) = D_{\varphi+(n+1)}(\tau).
\end{aligned}$$

Hence, by the definition,

$$\begin{aligned}
D_{\varphi+\omega}(\tau*\psi) &= \bigcap_{\alpha<\varphi+\omega} D_\alpha(\tau*\psi) = \bigcap_{n=0}^{\infty} D_{\varphi+2n}(\tau*\psi)\\
&\subset \bigcap_{n=0}^{\infty} D_{\varphi+n}(\tau) = D_{\varphi+\omega}(\tau).
\end{aligned}$$

In an analogous way we prove that $D_{\varphi+\omega}(\tau*\psi) \subset D_{\varphi+\omega}(\psi)$, thus

$$D_{\varphi+\omega}(\tau*\psi) \subset D_{\varphi+\omega}(\tau) \cap D_{\varphi+\omega}(\psi).$$

Now taking into account again the intersection definition of the sets $D_\alpha$ for limit numbers $\alpha$, and again using the induction we get the required conclusion (4).

If $f\colon X \to Y$ is a surjection between compact spaces $X$ and $Y$, and $\tau = \{X_\alpha\}_{\alpha<\Gamma}$ is a covering sequence of $X$, then the covering sequence $\{f(X_\alpha)\}_{\alpha<\Gamma}$ of $Y$ will be denoted by $f(\tau)$. The proof of the following is quite similar to the proof of Lemma 17(a) of **[P1]**, so it is omitted.

(5) If $\tau$ is a covering sequence of a compact space $X$ and $f\colon X \to Y$ is a surjection, then

$$D_\alpha(f(\tau)) \subset f(D_\alpha(\tau)) \quad \text{for every } \alpha < \Gamma.$$

Let a class $\mathscr{F}$ of compact spaces be invariant with respect to continuous mappings, i.e., we assume that each continuous image of any member of $\mathscr{F}$ belongs to $\mathscr{F}$. Denote by $\mathscr{F}_1$ the class of all compact spaces being countable unions of their compact subsets belonging to $\mathscr{F}$. Then, of course, each element of $\mathscr{F}_1$ admits covering sequences composed of elements of $\mathscr{F}$. Let $X \in \mathscr{F}_1$. Fix a limit ordinal $\Gamma$ with cardinality greater than card $X$. Further, put

$$S(X) = \{\tau\colon \tau = \{X_\alpha\}_{\alpha<\Gamma} \text{ is a covering sequence of } X \text{ with } X_\alpha \in \mathscr{F} \text{ for every } \alpha < \Gamma\}.$$

Then for every $\alpha < \Gamma$ we put

$$W_\alpha(X, \mathscr{F}) = \bigcap\{D_\alpha(\tau)\colon \tau \in S(X)\}.$$

We need the following two properties of these sets.

(6) There is the greatest limit number $\varphi_0 < \Gamma$ such that $W_{\varphi_0}(X, \mathscr{F}) \neq \varnothing$.

In fact, the sets $W_\alpha(X, \mathscr{F})$ form a decreasing sequence of compact subsets of $X$ with some $W_\alpha(X, \mathscr{F}) = \varnothing$ (see (3)). By the definitions we have $W_\varphi(X, \mathscr{F}) = \bigcap\{W_\alpha(X, \mathscr{F})\colon \alpha < \varphi\}$ for limit ordinals $\varphi$. Thus the number $\alpha_0 = \min\{\alpha\colon W_\alpha(X, \mathscr{F}) = \varnothing\}$ is nonlimit, so $\alpha_0 = \varphi_0 + n$, where $\varphi_0$ is limit and $n > 0$ is natural. The number $\varphi_0$ satisfies the required condition.

(7) There is a covering sequence $\tau \in S(X)$ with $\lambda(\tau) = \varphi_0 + n$ for some natural $n > 0$, where $\varphi_0$ is as in (6).

Indeed, since $\varnothing = W_{\varphi_0+\omega}(X, \mathscr{F}) = \bigcap\{D_{\varphi_0+\omega}(\tau)\colon \tau \in S(X)\}$ and since the sets $D_\alpha(\tau)$ are compact, there is a finite system $\tau_1, \dots, \tau_m \in S(X)$ with $\bigcap\{D_{\varphi_0+\omega}(\tau_i)\colon i \in \{1, \dots, m\}\} = \varnothing$. Put $\tau = (\cdots(\tau_1 * \tau_2) * \cdots * \tau_{m-1}) * \tau_m$. Then $D_{\varphi_0+\omega}(\tau) \subset \bigcap\{D_{\varphi_0+\omega}(\tau_i)\colon i \in \{1, \dots, m\}\} = \varnothing$ by (4). Hence $\lambda(\tau) \leqslant \varphi_0 + \omega$. But $\lambda(\tau)$ is always nonlimit (it may be observed by the definition), and $\varnothing \neq W_{\varphi_0}(X, \mathscr{F}) \subset D_{\varphi_0}(\tau)$, thus $\lambda(\tau) = \varphi_0 + n$ for some natural $n > 0$.

2. Lemma. *If $X \in \mathscr{F}_1$ and $f\colon X \to Y$ is a surjection, then for each limit ordinal $\varphi < \Gamma$ we have*

$$W_\varphi(Y, \mathscr{F}) \subset f(W_\varphi(X, \mathscr{F})).$$

Proof. Let $y \notin f(W_\varphi(X, \mathscr{F}))$, i.e.,

$$\varnothing = f^{-1}(y) \cap W_\varphi(X, \mathscr{F}) = f^{-1}(y) \cap \bigcap\{D_\varphi(\tau)\colon \tau \in S(X)\}$$
$$= \bigcap\{f^{-1}(y) \cap D_\varphi(\tau)\colon \tau \in S(X)\}.$$

Since all of the sets $f^{-1}(y) \cap D_\varphi(\tau)$ are closed subsets of the compact space $X$, there is a finite system $\tau_1, \ldots, \tau_m \in S(X)$ such that $\bigcap\{f^{-1}(y) \cap D_\varphi(\tau_i): i \in \{1, \ldots, m\}\} = \varnothing$. Put $\tau = (\cdots(\tau_1 * \tau_2) * \cdots * \tau_{m-1}) * \tau_m$. Then $\tau \in S(X)$, and by (4) we get

$$f^{-1}(y) \cap D_\varphi(\tau) \subset f^{-1}(y) \cap D_\varphi(\tau_1) \cap \cdots \cap D_\varphi(\tau_m) = \varnothing,$$

hence $y \notin f(D_\varphi(\tau))$. By (5) and by the definition of $W_\varphi(Y, \mathscr{F})$, noting that $f(\tau) \in S(Y)$, we see $W_\varphi(Y, \mathscr{F}) \subset D_\varphi(f(\tau)) \subset f(D_\varphi(\tau))$. Thus $y \notin W_\varphi(Y, \mathscr{F})$, which completes the proof.

The following lemma plays a crucial role in the proof of Theorem 1.

3. Lemma. *If $C_1, C_2, \ldots$ is a countable sequence of compact subsets of a compact space $X$ such that $X = C_1 \cup C_2 \cup \cdots$, then there is a compact nonempty subset $F$ of $X$ such that*

(1) *$F \subset f(F)$ for every surjection $f: X \to X$, and*

(2) *there are a finite sequence $C_{i_1}, \ldots, C_{i_n}$ of sets and a finite sequence of mappings $f_k: C_{i_k} \to X$ for $k \in \{1, \ldots, n\}$, such that $F \subset f_1(C_{i_1}) \cup \cdots \cup f_n(C_{i_n})$.*

Proof. Let $\mathscr{F}$ be the class of all continuous images of the sets $C_1, C_2, \ldots$. Then, of course, $\mathscr{F}$ is invariant with respect to continuous mappings and $X \in \mathscr{F}_1$. Put $F = W_{\varphi_0}(X, \mathscr{F})$, where $\varphi_0$ is the number guaranteed by (6). Then by Lemma 2 we get $F \subset f(F)$ for each surjection $f: X \to X$. Let $\tau = \{X_\alpha\}_{\alpha < \Gamma}$ be a covering sequence guaranteed by (7). Then since $D_{\varphi_0 + n}(\tau) = \varnothing$ we have

$$F = W_{\varphi_0}(X, \mathscr{F}) \subset D_{\varphi_0}(\tau) \subset X_{\varphi_0} \cup X_{\varphi_0+1} \cup \cdots \cup X_{\varphi_0+(n-1)},$$

and $X_{\varphi_0+i} \in \mathscr{F}$ for every $i \in \{0, 1, \ldots, n-1\}$, thus the sets $X_{\varphi_0+i}$ are continuous images of some sets $C_k$, which completes the proof.

Proof of Theorem 1. Let $F \subset X$ be a set guaranteed by Lemma 3 for $C_i = \operatorname{cl} A_i$. By (2) of Lemma 3 the set $F$ may be covered by finitely many subcontinua of $X$ each containing a dense $\mathscr{K}$-connected subset. Let $F_1, \ldots, F_m$ be such sets with an additional assumption that $m$ is the minimum number. Without loss of generality we may assume that $F_i = \operatorname{cl} B_i$ for some $\mathscr{K}$-components $B_i$ of $X$ for $i \in \{1, \ldots, m\}$.

Let $\mathscr{A}$ be the family of all $\mathscr{K}$-components of $X$.

(8) For every $A \in \mathscr{A}$ we have $\operatorname{cl} A \cap F \neq \varnothing$.

In fact, consider any surjection $f: X \to X$ sending a point of $B_1$ to a point of $A$. Then $f(B_1) \subset A$ and $f(F_1) = f(\operatorname{cl} B_1) \subset \operatorname{cl} A$. If $\operatorname{cl} A \cap F$ were empty, the union $f(F_2) \cup \cdots \cup f(F_m)$ of $m - 1$ continua with dense $\mathscr{K}$-connected subsets would contain $F$ (since $F \subset f(F) \subset f(F_1) \cup \cdots \cup f(F_m)$ and $f(F_1) \cap F \subset \operatorname{cl} A \cap F = \varnothing$), contrary to the assumption on $m$.

Put $\mathscr{A}_i = \{A \in \mathscr{A}: \operatorname{cl} A \cap F_i \neq \varnothing\}$ and $G_i = \operatorname{cl}(\bigcup \mathscr{A}_i)$ for $i \in \{1, \ldots, m\}$. We prove that

(9) $G_i \cap G_j = \varnothing$ for $i \neq j$, $i, j \in \{1, \ldots, m\}$.

Suppose $x \in G_i \cap G_j$ with $j > i$. Let $U$ be the nonempty interior of a $\mathscr{K}$-component $B_{m+1}$ of $X$ and let $f: X \to X$ be a surjection sending $x$ to a point $y \in U$. Thus $f(A), f(B) \subset B_{m+1}$ for some $A \in \mathscr{A}_i$ and $B \in \mathscr{A}_j$, and $\operatorname{cl} B_{m+1} \cap f(F_i) \neq \varnothing \neq \operatorname{cl} B_{m+1} \cap f(F_j)$. Let a surjection $g: X \to X$ send a point $p \in \operatorname{cl} B_{m+1} \cap f(F_i)$ to $y$.

Then $g(B_{m+1})$, $gf(B_i) \subset B_{m+1}$ and $\operatorname{cl} B_{m+1} \cap gf(F_j) \neq \varnothing$. Let a surjection $h$: $X \to X$ send a point $q \in \operatorname{cl} B_{m+1} \cap gf(F_j)$ to $y$. Then $h(B_{m+1})$, $hgf(B_j) \subset B_{m+1}$. Therefore

$$\begin{aligned} F \subset hgf(F) &\subset hgf(F_1) \cup \cdots \cup hgf(F_m) \\ &\subset hgf(F_1) \cup \cdots \cup hgf(F_{i-1}) \cup hgf(F_{i+1}) \cup \cdots \cup hgf(F_{j-1}) \\ &\quad \cup hgf(F_{j+1}) \cup \cdots \cup hgf(F_m) \cup \operatorname{cl} B_{m+1}. \end{aligned}$$

Thus $m - 1$ sets with dense $\mathscr{K}$-connected subsets cover $F$, contrary to the assumption on $m$.

The statement (9), by (8) and by the connectedness of $X$, implies

(10) $$m = 1.$$

(11) For every $A \in \mathscr{A}$ we have $F \subset \operatorname{cl} A$.

For, let a surjection $g$: $X \to X$ send a point of $B_1$ to a point of $A$. Then $F \subset g(F) \subset g(\operatorname{cl} B_1) = \operatorname{cl} g(B_1) \subset \operatorname{cl} A$.

(12) For every $A \in \mathscr{A}$ we have $\operatorname{cl} A = X$.

Indeed, for a given point $x \in X$ let a surjection $f$: $X \to X$ send a point $y \in F$ to $x$, and let $B \in \mathscr{A}$ be a $\mathscr{K}$-component of $X$ such that $f(B) \subset A$. Then $y \in \operatorname{cl} B$ by (11). Therefore $x = f(y) \in f(\operatorname{cl} B) = \operatorname{cl} f(B) \subset \operatorname{cl} A$. Thus we have (12).

To make the proof of Theorem 1 complete, note that since every $\mathscr{K}$-component of $X$ is dense in $X$ and one $\mathscr{K}$-component has nonempty interior, this $\mathscr{K}$-component is the only one.

**Applications and questions.** As an immediate consequence of Theorem 1 and of the Baire theorem we have the following corollary.

4. COROLLARY. *If a continuously homogeneous continuum is the countable union of arcwise connected ($\mathscr{K}$-connected) continua, then it is arcwise connected ($\mathscr{K}$-connected).*

5. THEOREM. *If a continuously homogeneous nondegenerate metric continuum $X$ contains no simple triod and it has only countably many arc components, then $X$ is either an arc or a simple closed curve.*

PROOF. Let $A$ be an arc component of $X$. Then one of the following statements is true:

(1) $A$ is degenerate,

(2) $A$ is a simple closed curve,

(3) $A$ is nondegenerate and it contains no simple closed curve.

In fact, note that if $A$ contains a simple closed curve, then $A$ itself is a simple closed curve (otherwise $A$ would contain a simple triod).

In case (3), since $A$ contains no simple closed curve, for all points $a, b \in A$ with $a \neq b$ there is only one arc $ab$ in $A$. Further we observe that in this case one of the following is true:

(3.1) $A$ has two end points,

(3.2) $A$ has one end point,

(3.3) $A$ has no end point.

Namely, if $e_1, e_2 \in A$ are distinct end points of $A$, then each point $p \in A$ belongs to $e_1e_2$. Indeed, if not, let $q$ be the first point of the arc $pe_1$ lying in the arc $e_1e_2$. Then $pq \cup qe_1 \cup qe_2$ is a simple triod for $q \neq e_1$ and $q \neq e_2$. Thus (3.1)–(3.3) are all possibilities and in case (3.1) $A$ is an arc.

In case (3.2) let $e$ be the end point of $A$. Then we inductively construct a well-ordered sequence $\{A_\alpha\}$ of arcs contained in $A$ with $\bigcup_\alpha A_\alpha = A$. Namely,

$$A_0 = ep \quad \text{for a point } p \in A \setminus \{e\},$$

$$A_\alpha = eq \quad \text{for a point } q \in A \setminus \bigcup\{A_\beta \colon \beta < \alpha\}, \text{ for } \alpha > 0$$

(if such $q$ exists). Let $A_\alpha = ex$ and $A_\beta = ey$ for $\alpha > \beta$. We have $x \notin ey$, thus $ex \not\subset ey$, so $ey \subsetneqq ex$. Hence the sequence $\{A_\alpha\}$ is increasing, thus countable. Further, we may observe that it is a one-to-one image of the half-line.

Similarly we prove that in case (3.3) $A$ is a one-to-one image of the real line (details of this proof are left to the reader).

Each of these cases implies that $A$ is an $F_\sigma$-set, thus each arc component of $X$ is an $F_\sigma$-set. Since $X$ has countably many arc components only, by the Baire theorem, we infer that at least one of them has nonempty interior. Applying Theorem 1 we see that $X$ is arcwise connected. Thus $X$ is the only arc component satisfying either (2) or (3). Suppose it is neither an arc nor a simple closed curve. Thus it is nonlocally connected (since nondegenerate atriodic locally connected continuum is either an arc or a simple closed curve). There are two possibilities only: (3.2) and (3.3), i.e., $X$ is a one-to-one image either of the half-line or of the real line. But Krupski showed in [**K1**, Theorem 4, p. 352] that compact nonlocally connected one-to-one images of the half-line or of the real-line are not continuously homogeneous. This contradiction completes the proof.

J. J. Charatonik and T. Maćkowiak posed in [**CM**, Problem 3.11] the problem of characterizing continuously homogeneous arc-like continua. The former of the following two corollaries may be considered as a step in a way to do it. It also improves Corollary 1 of [**K1**, p. 354].

6. COROLLARY. *Let $X$ be a nondegenerate continuously homogeneous metric continuum with only countably many arc components. Then the following statements are equivalent*:

(a) *$X$ is an arc*,

(b) *$X$ is arc-like*,

(c) *$X$ contains neither a simple triod nor a simple closed curve*.

7. COROLLARY. *Under the same assumptions as in Corollary* 6, *the following statements are equivalent*:

(a) *$X$ is a simple closed curve*,

(b) *$X$ is circle-like*,

(c) *$X$ is not an arc and it contains no simple triod*.

There are some interesting questions closely related to the subject of this paper, and also to the results of [**K1, K2, CG**].

QUESTION 1. If a continuum $X$ is continuously homogeneous and has $\mathscr{K}$-components (arc components) $A_1, A_2, \ldots$ with $X = \mathrm{cl}\, A_1 \cup \mathrm{cl}\, A_2 \cup \cdots$, is each $\mathscr{K}$-component (arc component) of $X$ necessarily dense in $X$?

We know that each arc component of a continuously homogeneous continuum $X$ with finitely many arc components is dense in $X$ (see Theorem 3 of **[CG]**).

QUESTION 2. If a continuously homogeneous continuum $X$ has only countably many $\mathscr{K}$-components (arc components), is each $\mathscr{K}$-component (arc component) necessarily dense in $X$?

QUESTION 3. Under the same conditions as in Question 2, is $X$ necessarily $\mathscr{K}$-connected (arcwise connected)?

QUESTION 4. What about an answer to Question 3 if we additionally assume that $X$ has only a finite number of $\mathscr{K}$-components (arc components)?

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