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ON 3-MANIFOLDS HAVING SURFACE-BUNDLES AS BRANCHED COVERINGS

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To Professor F. Botella on his seventieth birthday

ABSTRACT. We give a different proof of the result of Sakuma that every closed, oriented 3-manifold M has a 2-fold branched covering space N which is a surface bundle over S^1. We also give a new proof of the result of Brooks that N can be made hyperbolic. We give examples of irreducible 3-manifolds which can be represented as $2m$-fold cyclic branched coverings of S^3 for a number of different m's as big as we like.

1. In [S] Sakuma proves that *for every closed, oriented, connected 3-manifold M^3, there exists an F_g-bundle over S^1, W^3, where F_g is a closed, oriented and connected surface of genus g, such that W^3 is a 2-fold branched covering of M^3.*

He shows this by thinking of a handlebody X_g, of genus g, as the mapping cylinder of $f\colon F_g \to P_g$ (defined in Figure 1).

If M has a Heegaard splitting $M = X_g \cup X'_g$, there is a 2-fold covering of M branched over $\partial P_g \cup \partial P'_g$ that can be constructed by splitting M along $P_g \cup P'_g$ and pasting together two copies of the resulting F_g-bundle over [0,1]. This 2-fold covering is W^3.

2. In this note we give an alternative proof of the same result. Namely, we show

LEMMA 1. *Let M^3 be a closed, oriented 3-manifold having an open book structure, $M^3 = M(F_{g,h};\phi)$, where $F_{g,h}$ is a compact, connected, and oriented surface of genus g with h boundary components, and where $\phi\colon F_{g,h} \to F_{g,h}$ (the monodromy map) is a homeomorphism which restricts to the identity map on the boundary of $F_{g,h}$. Then there exists an F_k-bundle over S^1, $M(\phi \# \phi^{-1})$, which is a 2-fold covering of $M(F_{g,h};\phi)$ branched over a $2h$-component link, and where $k = 2g + h - 1$.*

Since every M^3 is an open book $M(F_{g,1};\phi)$ [**GA, M**] we deduce

COROLLARY 2. *Every closed, oriented connected 3-manifold M^3 contains a 2-component link L, such that there is a 2-fold covering of M^3 branched over L which is an F_k-bundle over S^1.*

PROOF OF LEMMA 1. Let $2F_{g,h}$ be the double of $F_{g,h}$ and let $i\colon 2F_{g,h} \to 2F_{g,h}$ be the natural involution interchanging the two copies of $2F_{g,h}$.

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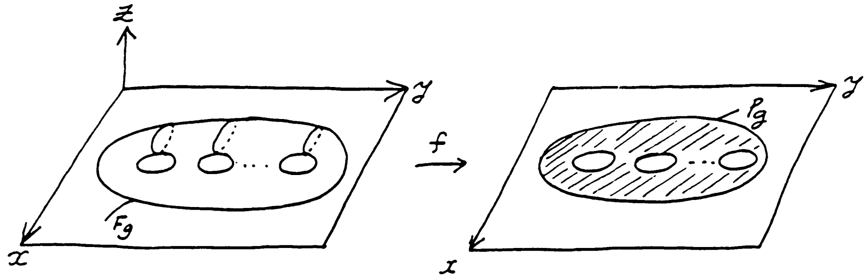

FIGURE 1. f projects F_g onto the (x, y)-plane

Let $M(\phi \# \phi^{-1})$ be the $2F_{g,h}$-bundle over S^1 with monodromy $\phi \# \phi^{-1}$ defined by

$$\frac{2F_{g,h} \times [-1,1]}{(x,1) \equiv (\phi \# \phi^{-1}x, -1)},$$

where $\phi \# \phi^{-1}\colon 2F_{g,h} \to 2F_{g,h}$ is ϕ in one copy of $F_{g,h}$, and is ϕ^{-1} in the other copy.

Consider the involution $u\colon M(\phi \# \phi^{-1}) \to M(\phi \# \phi^{-1})$ given by $u(x,t) = (ix, -t)$ for every $(x,t) \in 2F_{g,h} \times [-1,1]$. It has a pair of double curves for each component of $\partial F_{g,h}$, namely $\operatorname{Fix} u = \partial F_{g,h} \times \{0,1\}$.

The quotient of $M(\phi \# \phi^{-1})$ under u is the quotient of

$$\frac{F_{g,h} \times [-1,1]}{(x,1) \equiv (\phi x, -1)}$$

under the identification $(x,t) \equiv (x,-t)$ for every

$$(x,t) \in (\partial F_{g,h} \times [-1,1])/((x,1) \equiv (x,-1)).$$

This is equivalent to identifying $x \times [0,1]$ with $x \times [0,-1]$, for every $x \in \partial F_{g,h}$, or to collapsing $x \times S^1$ into a point for every $x \in \partial F_{g,h}$. Therefore, the quotient of $M(\phi \# \phi^{-1})$ under the action of u is the open book $M(F_{g,h};\phi)$. $\square$

REMARK. The branching set of the 2-fold covering of the lemma is the boundary of a collar of a leaf.

EXAMPLE. The figure-eight knot 4_1 in S^3 is a fibered knot. The 2-fold covering of S^3 branched over the union of 4_1 and its canonical longitude (i.e. the boundary of a collar of a fiber) is an F_2-bundle over S^1 with monodromy $\phi \# \phi^{-1}$, where $\phi = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ on a torus with a hole. This 2-fold covering is the result of 0-Dehn surgery on $4_1 \# 4_1$, and this representation helps to visualize the F_2-bundle structure. As a consequence of this example, note that the m-cyclic covering of S^3 branched over 4_1 is the quotient of $M(\phi^m \# \phi^{-m})$ under the action of u (defined in the proof of the lemma).

Now consider the link $L_r(F_{g,h};\phi)$ in $M(F_{g,h};\phi)$ formed by the boundary of a collar C of a leaf of $M(F_{g,h};\phi)$ together with the union of r sections of $M(F_{g,h};\phi)$ not intersecting C (see $L_3\left(F_{1,1}; \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]\right)$ in Figure 2). For each $m > 1$ there is a regular covering of $M(F_{g,h};\phi)$, branched over $L_r(F_{g,h};\phi)$, with group of covering translations $\mathbf{Z}_2 \times \mathbf{Z}_m$ which is the composition of the 2-fold covering

$$p\colon M(\phi \# \phi^{-1}) \longrightarrow M(F_{g,h};\phi)$$

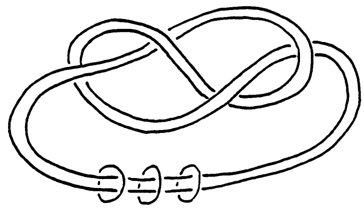

FIGURE 2

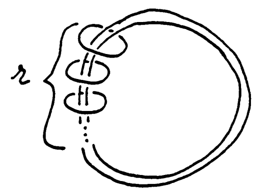

FIGURE 3

defined in the proof of the lemma, with an m-cyclic covering

$$q\colon W \longrightarrow M(\phi \# \phi^{-1})$$

branched over $p^{-1}(L_r(F_{g,h};\phi)\backslash\partial C)$. The manifold W is an F_k-bundle over S^1, and the genus k of the fiber F_k increases as m or r increases. Thus

COROLLARY 3. *Every closed, oriented 3-manifold admits $2m$-fold cyclic branched coverings, m = odd, which are F_k-bundles over S^1, and where m and k are as big as we like.* □

REMARK. This observation could have been easily obtained using the method of Sakuma [**S**], by considering Heegaard splittings of arbitrarily big genus. But with our method we gain control on the branching set, as we see in the next example.

EXAMPLE. The $2m$-fold branched covering corresponding to $L_r(F_{0,0};\mathrm{id})$ is $S^1 \times F_h$, where $h = (m-1)(r-1)$, $m > 1$, $r \geq 0$ (Figure 3). This is a *cyclic* covering if and only if m =odd. Thus $S^1 \times F_2$ is a 6-fold cyclic covering of S^3 [**HN**]. More generally:

COROLLARY 4. *$S^1 \times F_{2^x}$ is a cyclic branched covering of S^3 of $2(2^y+1)$ sheets, for every $y \leq x$.*

Thus we see that *there are irreducible 3-manifolds which can be represented as $2m$-fold cyclic branched coverings of S^3 for a number of different m's as big as we like.*

We end this paper with three questions.

QUESTION 1. *Are there closed, orientable, irreducible 3-manifolds which are m-fold cyclic branched coverings of S^3 for as many primes m as we like?*

Since the branching sets $L_r(F_{0,0};\mathrm{id})$ have bridge number $r+2$, one would expect that the answer to the next question would be in the affirmative.

QUESTION 2. *Does $S^1 \times F_{2^x}$ have at least x different minimal Heegaard splittings (i.e. no trivial handles) all of them of different genus?* (Compare [**CG**].)

Corollary 3 suggests the next question.

QUESTION 3. *Is there a prime $p \neq 2$ such that every closed, oriented 3-manifold has a branched p-fold cyclic covering which is an F_g-bundle over S^1?*

ADDED IN JULY 1986. After this paper was accepted I have seen [**B**] where, using Sakuma's method, it is shown that every closed, oriented connected 3-manifold has a 2-fold branched covering which is a *hyperbolic manifold* and an F_g-bundle over S^1. The same result can be obtained using the methods of this paper. To see this, we follow the notation of Lemma 1. First take $F_{g,1}$ to be the fiber of a hyperbolic fibered knot $K = \partial F_{g,1}$ in M^3 [**So**]. Then the boundary $K \cup K'$ of a collar A of $F_{g,1}$ is the branching set of a 2-fold covering $M(\phi \# \phi^{-1})$ where ϕ is the monodromy of K. Since ϕ is pseudo-Anosov there exists a simple arc $\hat{\gamma}$ properly embedded in $F \setminus \operatorname{Int} A$ (i.e. $\partial\hat{\gamma} \subset K'$) such that the orbit of $\hat{\gamma}$ under ϕ fills $F \setminus \operatorname{Int} A$ [**F**]. Modifying K' suitably in a regular neighborhood of $\hat{\gamma}$ changes the 2-fold covering $M(\phi \# \phi^{-1})$ by $\frac{1}{n}$-Dehn surgery on γ covering $\hat{\gamma}$ [**Mo**]. Here γ is a simple closed curve which doubles $\hat{\gamma}$ in $2F_{g,1}$, the fiber of $M(\phi \# \phi^{-1})$. The manifold resulting from this $\frac{1}{n}$-Dehn surgery on $M(\phi \# \phi^{-1})$ still is a 2-fold covering of M, branched over $K\cup$ (K modified along $\hat{\gamma}$), and a $2F_{g,1}$-bundle over S^1, but the monodromy is the composition of $\phi \# \phi^{-1}$ with T_γ^n, the nth power of a Dehn twist along γ [**St**]. Since the orbit of γ under $\phi \# \phi^{-1}$ fills $2F_{g,1}$, it follows from [**F**] that $T_\gamma^n(\phi \# \phi^{-1})$ is pseudo-Anosov except for at most seven consecutive values of n. This finishes the proof of the theorem.

The last proof was obtained with the generous help of F. Bonahon and M. Boileau. It is a pleasure to record here my warmest thanks to both of them.

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