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THE SIMPLICIAL BUNDLE OF A CW FIBRATION

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ABSTRACT. Suppose we are given a fibration $p\colon E \to B$ over a connected base, with both base and fibre having the homotopy type of CW complexes. We construct a fibre bundle over B fibre homotopy equivalent to the given fibration and with fibre a simplicial complex. Further, the transformations of the fibre arising from the transition functions of this bundle are simplicial maps. From this, we deduce that the weak spectral sequence constructor axioms are sufficient to determine the Serre spectral sequence of a CW fibration.

In Barnes [**1**], it was shown that a certain set of axioms determines the Serre spectral sequence of a CW fibration. It was conjectured that a weaker set of axioms would suffice, but this was only proved for simplicial bundles. In this paper, we extend this to arbitrary CW fibrations by showing how to construct a simplicial bundle fibre homotopy equivalent to a given CW fibration.

The construction follows those of Milnor [**4**] and Fadell [**3**]. In order to ensure that the space S of simplicial paths in the triangulated space B is a CW complex, they restricted B to being a countable or locally finite complex. We shall also need such a space S to be a CW complex, but our intended application does not permit us to impose a finiteness condition on B. Instead, we change the topology on S, using the compactly generated space $k(S)$ associated with S. (See Steenrod [**5**].) This enables us to use Fadell's construction, but we lose the topology on the group of the bundle. The group G of simplicial loops acts on the fibres, but it is not clear that it can be topologised so as to satisfy all the requirements for the structure group of a Steenrod fibre bundle. For convenience, we shall refer to G as the structure group and call various bundles G-bundles, but G should be understood to be a group without topology.

In our construction, we shall make use of a functor s from topological spaces to simplicial complexes, and of a natural transformation $j\colon s \to$ identity with the property that $j_X\colon s(X) \to X$ is a homotopy equivalence whenever X has the homotopy type of a CW complex. Such a functor may be obtained by taking $s(X)$ to be the second barycentric subdivision of the geometric realisation of the singular complex of X, with j_X the evaluation map.

THEOREM. *Suppose B has the homotopy type of a connected CW complex. Then there exists a group G (not topologised) and a functor $\mathcal{B}$ from the category of fibrations over B with fibre having the homotopy type of a CW complex, to the category of G-bundles over B such that, for each such fibration $p\colon E \to B$,*

(i) *$\mathcal{B}(p)\colon \mathcal{B}(E) \to B$ is fibre homotopy equivalent to p,*

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(ii) *the fibre of* $\mathcal{B}(p)$ *is a simplicial complex, and*
(iii) G *acts on the fibre by simplicial maps.*

PROOF. Let $p\colon E \to B$ be a fibration with some (and hence every) fibre having the homotopy type of a CW complex. We proceed in a number of stages, constructing various auxilliary spaces and fibrations.

Step 1. *Replace* B *with a simplicial complex.* Put $C = s(B)$ and form the pullback $j_B^*(p) = q\colon K \to C$ of p along the evaluation map $j_B\colon s(B) \to B$. Then C is a simplicial complex and $q\colon K \to C$ is a fibration whose fibres are of the same homotopy type as those of p.

$$\begin{tikzcd} E & K \arrow[l] \arrow[d, "q=j_B^*(p)"] \\ B \arrow[from=1-1, "p"] & C = s(B) \arrow[l, "j_B"] \end{tikzcd}$$

Step 2. *Replace* $K \to C$ *with a fibre bundle.* We follow the construction of Fadell [**3**], amended to allow for C not being locally finite. Let S_n be the set of all sequences $(x_n, x_{n-1}, \ldots, x_0) \in C^{n+1}$ such that each pair x_i, x_{i-1} lie in a common simplex of C. We topologise S_n with the topology induced from $k(C^{n+1})$, that is, from the weak topology on C^{n+1} given by the product cells. Let S_* be the disjoint union of the S_n for $n \geq 1$. Let $\sim$ be the equivalence relation in S_* generated by the relations

$$(x_n, \ldots, x_{i+1}, x_i, x_{i-1}, \ldots, x_0) \sim (x_n, \ldots, x_{i+1}, x_{i-1}, \ldots, x_0)$$

whenever $x_i = x_{i-1}$ or $x_{i+1} = x_{i-1}$. The space S of simplicial paths in C is defined to be $S_*/{\sim}$. It follows as in Milnor [**4**] that S is a CW complex. We denote the equivalence class of $(x_n, \ldots, x_0)$ by $[x_n, \ldots, x_0]$. For the simplicial path $\alpha = [x_n, \ldots, x_0]$, we set $\alpha(0) = x_0$ and $\alpha(1) = x_n$.

We now select a vertex v of C. We define G to be the group of simplicial loops at v. It need not be a topological group. Milnor's argument only shows that multiplication gives a continuous functions $k(G \times G) \to G$. Thus G is a group in the category of compactly generated spaces. We ignore the topology on G.

We put

$$L = \{(\kappa, \alpha) \in K \times S \mid q(\kappa) = \alpha(0)\}$$

with topology induced by the product topology of $K \times S$, and define $r(\kappa, \alpha) = \alpha(1)$. As in Fadell [**3**], it follows that $r\colon L \to C$ is a fibre bundle with structure group (in our limited sense) G, and that the map $\lambda\colon K \to L$ given by $\lambda(\kappa) = (\kappa, [q(\kappa), q(\kappa)])$ is a fibre homotopy equivalence. Note that Fadell's argument requires that S and $\{\alpha \in S \mid \alpha(0) = v\}$ be CW complexes. It does not require L to be a CW complex.

$$\begin{tikzcd} E \arrow[d, "p"] & K \arrow[l] \arrow[r, "\lambda"] \arrow[d, "q"] & L \arrow[d, "r"] \\ B & C \arrow[l, "j_B"] \arrow[r, "\mathrm{id}"] & C \end{tikzcd}$$

Step 3. *Replace* L *with a simplicial bundle.* We now construct a simplicial bundle $t\colon M \to C$. For each $c \in C$, we define $M_c = s(L_c)$ where $L_c = r^{-1}(c)$. We put

$M = \bigcup_{c \in C} M_c$ and define the function $t\colon M \to C$ by $t(M_c) = c$. We still have to define the topology of M. Now $r\colon L \to C$ is a bundle. For some covering of C by open sets U_α, we have coordinate functions $\phi_\alpha\colon U_\alpha \times F \to r^{-1}(U_\alpha)$, where F is the fibre over the vertex v. Let $\Phi = s(F)$. For each $c \in U_\alpha$, the map $\phi_{\alpha,c} = \phi_\alpha \mid c \times F \to L_c$ induces a map $\psi_{\alpha,c} = s(\phi_{\alpha,c})\colon c \times \Phi \to M_c$. Putting these together, we have a function $\psi_\alpha\colon U_\alpha \times \Phi \to t^{-1}(U_\alpha)$. We define the open sets of $t^{-1}(U_\alpha)$ to be the images under ψ_α of the open sets of $U_\alpha \times \Phi$, thus making ψ_α a homeomorphism. Since $r\colon L \to C$ is a bundle, this definition is consistent on intersections $t^{-1}(U_\alpha \cap U_\beta)$ and clearly makes $t\colon M \to C$ a G-bundle. We have in effect built up M by patching together pieces $U_\alpha \times \Phi$ using transition functions $\psi_\beta^{-1}\psi_\alpha$ induced by the $\phi_\beta^{-1}\phi_\alpha$. Since L is fibre homotopy equivalent to K, each fibre L_c has the homotopy type of a CW complex. Thus the evaluation maps $M_c = s(L_c) \to L_c$ are homotopy equivalences. Taken together, these maps define a function $\mu\colon M \to L$ which, by considering coordinate neighbourhoods U_α, is easily seen to be continuous. By Fadell [**2**], $\mu\colon M \to L$ is a fibre homotopy equivalence.

$$\begin{tikzcd}
E \arrow[d, "p"] & K \arrow[l] \arrow[r, "\lambda"'] \arrow[d, "q"] & L \arrow[d, "r"] & M \arrow[l, "\mu"] \arrow[d, "t"] \\
B & C \arrow[l, "j_B"] \arrow[r, "\mathrm{id}"'] & C & C \arrow[l, "\mathrm{id}"]
\end{tikzcd}$$

Step 4. *Pull back to* B. The map $j_B\colon C = s(B) \to B$ is a homotopy equivalence. We choose a homotopy inverse $f\colon B \to C$ of j_B. We define $\mathcal{B}(p)\colon \mathcal{B}(E) \to B$ to be the pullback along f of $t\colon M \to C$. Then $\mathcal{B}(p)$ is a G-bundle over B.

$$\begin{tikzcd}
E \arrow[d, "p"] & K \arrow[l] \arrow[r, "\lambda"'] \arrow[d, "q"] & L \arrow[d, "r"] & M \arrow[l, "\mu"] \arrow[d, "t"] & \mathcal{B}(E) \arrow[l] \arrow[d, "\mathcal{B}(p)"] \\
B & C \arrow[l, "j_B"] \arrow[r, "\mathrm{id}"'] & C & C \arrow[l, "\mathrm{id}"] & B \arrow[l, "f"]
\end{tikzcd}$$

In constructing $\mathcal{B}(p)$, we chose a vertex v of C and a map $f\colon C \to B$. These choices required knowledge only of the base B. We use the same vertex v and map $f\colon C \to B$ for all fibrations over B. Our construction is then clearly functorial, any map $E \to E'$ of fibrations over B giving rise to induced maps at all stages of the construction. On the G-bundles, these maps are clearly G-maps. It is easily seen that different choices of v and f give rise to naturally equivalent functors. We now verify the properties (i), (ii) and (iii).

We have $t\colon M \to C$ fibre homotopy equivalent to $j_B^*(p) = q\colon K \to C$. Hence $\mathcal{B}(p) = f^*(t)$ is fibre homotopy equivalent to $f^*j_B^*(p)$. But $j_Bf \sim \mathrm{id}\colon B \to B$. Thus $\mathcal{B}(p)$ is fibre homotopy equivalent to p.

By our construction, the fibre $\Phi = s(F)$ is a simplicial complex. An element $g \in G$ acts on F giving a homeomorphism $g_F\colon F \to F$. Since s is a functor to simplicial complexes, the induced map $g_\Phi = s(g_F)\colon \Phi \to \Phi$ is simplicial. $\square$

A spectral sequence constructor for the Serre spectral sequence is a functor $\mathcal{F}$ from CW fibrations $\pi\colon A \to B$ to filtered chain complexes satisfying certain axioms. Two versions of these axioms were discussed in Barnes [**1**, Chapter XIII]. For the convenience of the reader, we reproduce the weaker version.

HOMOTOPY AXIOM. If $f, g\colon \pi \to \pi'$ are homotopic, then $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are homotopic by a chain homotopy of filtration degree at most one. If f and g are fibre homotopic, then $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are homotopic by a chain homotopy of filtration degree zero.

TOTAL SPACE AXIOM. There is a natural isomorphism $\eta\colon H_\bullet \mathcal{F}(\pi) \to H_\bullet(A)$.

EXACTNESS AXIOM. If $f\colon \pi \to \pi'$ is injective, then $\mathcal{F}(f)\colon \mathcal{F}(\pi) \to \mathcal{F}(\pi')$ is injective and

$$(\mathcal{F}f)(\mathcal{F}^p\pi) = (\mathcal{F}f)(\mathcal{F}\pi) \cap \mathcal{F}^p(\pi').$$

EXCISION AXIOM. If $f\colon (\pi_1, \pi_1') \to (\pi, \pi')$ is an inclusion of fibration pairs, all fibrations over the same base, and the map $(A_1, A_1') \to (A, A')$ of pairs of total spaces is excisive, then the induced maps $E^1_{pq}\mathcal{F}(\pi_1, \pi_1') \to E^1_{pq}\mathcal{F}(\pi, \pi')$ are isomorphisms.

E^1 AXIOM. There exists a functor $C_\bullet$ from spaces with local coefficients to chain complexes which defines a local coefficient homology theory, and natural isomorphisms $\xi_\bullet\colon E^1_{\bullet 0}\mathcal{F}(\pi) \to C_\bullet(B, H_0(A_b))$.

FIBRE DIMENSION AXIOM. If A_b is discrete, then $E^1_{pq}\mathcal{F}(\pi) = 0$ for $q \neq 0$.

ADDITIVITY AXIOM. If $\pi_\alpha \subseteq \pi$, the A_α are open in A, and A is the disjoint union of the A_α, then $\mathcal{F}(\pi) = \bigoplus_\alpha \mathcal{F}(\pi_\alpha)$.

COROLLARY. *Suppose $\mathcal{F}, \mathcal{F}'$ satisfy the weak spectral sequence constructor axioms. Let $p\colon E \to E$ be a fibration in which B and the fibres $p^{-1}(b)$ have the homotopy type of CW complexes. Then $\mathcal{F}(p)$ and $\mathcal{F}'(p)$ have equivalent spectral sequences.*

PROOF. By the Additivity Axiom, we need only consider the case where B is connected. For any constructor, homotopy equivalent fibrations give equivalent spectral sequences. By the theorem, we may replace $p\colon E \to B$ by a simplicial bundle, and the result follows by Barnes [**1**, Theorems XIII.3.2 and XIII.4.4]. □

REFERENCES

1. D. W. Barnes, *Spectral sequence constructors in algebra and topology*, Mem. Amer. Math. Soc., vol. 53, no. 317, 1985.
2. E. Fadell, *On fiber homotopy equivalence*, Duke Math. J. **26** (1959), 699–706.
3. ———, *The equivalence of fiber spaces and bundles*, Bull. Amer. Math. Soc. **66** (1960), 50–53.
4. J. Milnor, *Construction of universal bundles.* I, Ann. of Math. (2) **63** (1956), 272–284.
5. N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152.

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