THE PREPARATION THEOREM FOR COMPOSITE FUNCTIONS

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ABSTRACT. We present a simple extension of the preparation theorem of B. Malgrange and J. Mather to the case of composite functions. As a corollary we obtain a short proof of the equivariant preparation theorem of V. Poénaru.

1. Formulation of the results. We denote by $\mathcal{E}(p)$ the ring of germs of $C^\infty$-functions at $0 \in \mathbb{R}^p$, by $\mathcal{E}(p, q)$ the set of germs at $0 \in \mathbb{R}^p$ of $C^\infty$-transformations $\mathbb{R}^p \to \mathbb{R}^q$ which preserve the origin, and by $m(p)$ the maximal ideal of $\mathcal{E}(p)$ formed by all functions vanishing at 0. A transformation $H \in \mathcal{E}(p, q)$ induces a local ring homomorphism $\mathcal{E}(q) \xrightarrow{H^*} \mathcal{E}(p)$ defined by $\beta \mapsto \beta \circ H$.

For $p \in \mathcal{E}(m, k)$ and $\eta \in \mathcal{E}(n, l)$ we introduce $\mathcal{E}_\rho(m) \overset{\text{def}}{=} \{ \alpha \circ \rho; \alpha \in \mathcal{E}(k) \}$, $\mathcal{E}_\eta(n) \overset{\text{def}}{=} \{ \beta \circ \eta; \beta \in \mathcal{E}(l) \}$, $m_\rho(m) \overset{\text{def}}{=} m(m) \cap \mathcal{E}_\rho(m)$, and $m_\eta(n) \overset{\text{def}}{=} m(n) \cap \mathcal{E}_\eta(n)$. Obviously $m_\eta(n) = \eta^* m(l)$ and $m_\rho(m) = \rho^* m(k)$.

A germ $f \in \mathcal{E}(m, n)$ such that $f^* \mathcal{E}_\eta(n) \subset \mathcal{E}_\rho(m)$ will be called a $\rho\eta$-germ; for such a transformation each component of $\eta \circ f$ belongs to $\mathcal{E}_\rho(m)$ and so is of the form $F_i \circ \rho$. Hence there exists $F \in \mathcal{E}(k, l)$ such that the following diagram commutes:

$$\begin{array}{ccc}
R^k & \xrightarrow{F} & R^l \\
\uparrow \rho & & \uparrow \eta \\
R^m & \xrightarrow{f} & R^n
\end{array}$$

As $\rho^* \mathcal{E}(k)$ and $f^* \mathcal{E}_\eta(n) = (\eta \circ f)^* \mathcal{E}(l) = (F \circ \rho)^* \mathcal{E}(l)$ are all subrings of $\mathcal{E}_\rho(m)$, any $\mathcal{E}_\rho(m)$-module $A$ could be considered as a $\mathcal{E}(k)$-, $\mathcal{E}_\rho(n)$- or $\mathcal{E}(l)$-module. Obviously $a_1, \ldots, a_p$ generate $A$ as an $\mathcal{E}_\rho(m)$-module (respectively an $\mathcal{E}_\eta(n)$-module) if and only if they generate it as an $\mathcal{E}(k)$-module ($\mathcal{E}(l)$-module, respectively). A similar remark concerns the generators of the isomorphic vector spaces

$$A/F^* m(l) \cdot A \approx A/(\eta \circ f)^* m(l) \cdot A \approx A/f^* m_\eta(n) \cdot A.$$

From the preparation theorem [1, p. 59, 2, 3] applied to $F$ we obtain the following result.

**Theorem 1.** Let $\rho \in \mathcal{E}(m, k)$, $\eta \in \mathcal{E}(n, l)$, let $A$ be a finitely generated $\mathcal{E}_\rho(m)$-module, let $f \in \mathcal{E}(m, n)$ be a $\rho\eta$-germ, and suppose $F \in \mathcal{E}(k, l)$ makes the diagram commute.
(1) commute. Then the elements \( a_1, \ldots, a_p \) generate \( A \) as an \( \mathcal{E}_n(m) \)-module if and only if they represent generators of the real vector space \( A/(f^*m_\eta(n) \cdot A) \).

Let \( m(\rho, k) \) denote the ideal in \( \eta(k) \) of all functions vanishing on \( \rho(\mathbb{R}^m) \). Note that the ring \( \mathcal{E}_\rho(m) \) is isomorphic to \( \mathcal{E}(k)/m(\rho, k) \). Hence for any \( \mathcal{E}(k) \)-module \( A^* \) the factor module \( A^*/(m(\rho, k) \cdot A^*) \) has the natural structure of an \( \mathcal{E}_\rho(m) \)-module.

**Corollary 1.** With the hypotheses of Theorem 1 let \( A \overset{\text{def}}{=} A^*/m(\rho, k) \cdot A^* \), where \( A^* \) is a finitely generated \( \mathcal{E}(k) \)-module. Then \( a_1, \ldots, a_p \) belonging to \( A^* \) represent generators of the \( \mathcal{E}_n(m) \)-module \( A \) if and only if they represent generators of the real vector space

\[
A^*/(m(\rho, k) \cdot A^* + F^*m(l) \cdot A^*).
\]

**Proof.** Let us denote by \( A_1 \) the \( \mathcal{E}(k) \)-submodule \( m(\rho, k) \cdot A^* \) of \( A^* \). From the following sequence of the natural \( \mathcal{E}(k) \)-module isomorphisms

\[
A/f^*m_\eta(n) \cdot A = A/\rho^*F^*m(l) \cdot A \cong A/F^*m(l) \cdot A \\
\cong (A_1/(F^*m(l) \cdot A^*)/A_1) \cong A^*/(F^*m(l) \cdot A^* + A_1),
\]

it follows that the vector spaces \( A/f^*m_\eta(n) \cdot A \) and \( A^*/(F^*m(l) \cdot A^* + m(\rho, k) \cdot A^*) \) are isomorphic. Now we can refer to Theorem 1, since \( A \) is a finitely generated \( \mathcal{E}_\rho(m) \)-module (because \( A^* \) is finitely generated over \( \mathcal{E}(k) \)).

**2. Equivariant division theorem.** This paragraph provides some examples of applications of Theorem 1.

Consider a compact Lie group \( G \) acting orthogonally on \( \mathbb{R}^m \) and \( \mathbb{R}^n \). According to G. Schwarz [5] there exist polynomial maps \( \rho \in \mathcal{E}(m, k) \) and \( \eta \in \mathcal{E}(n, l) \), called Hilbert maps, such that \( \mathcal{E}_\rho(m) \) and \( \mathcal{E}_\eta(n) \) are exactly the sets of \( G \)-invariant germs \( \mathcal{E}_G(m) \) and \( \mathcal{E}_G(n) \), respectively. Denote \( m_G(n) \overset{\text{def}}{=} m(n) \cap \mathcal{E}_G(n) \). Obviously any \( G \)-equivariant \( f \in \mathcal{E}(m, n) \) is a \( \rho \eta \)-germ, so from Theorem 1 there follows the equivariant preparation theorem [4].

**Theorem 2.** If \( A \) is a finitely generated \( \mathcal{E}_G(m) \)-module then \( A \) is finitely generated as a \( \mathcal{E}_G(n) \)-module if and only if the real vector space \( A/f^*m_G(n)A \) has a finite dimension.

**Example.** Let \( \mathbb{R}^m = \mathbb{R}^n = \mathbb{R}^2 \). Let \( G = Z_2 = \{ \pm 1 \} \) operate on \( \mathbb{R}^2 \) as \((x, y) \mapsto (\varepsilon x, \varepsilon y)\) for \( \varepsilon \in G \). Let us consider an equivariant transformation \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \mapsto (x, x^3 + y^3) \). Using Corollary 1 we shall show that \( 1, xy, y^2, y^4 \) generate \( \mathcal{E}_{Z_2}(2) \) over \( \mathcal{E}_{Z_2}(2) \).

The Hilbert maps \( \rho = \eta: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) could be defined as \((x, y) \mapsto (x^2, y^2, xy)\). The set \( \rho(\mathbb{R}^2) \subseteq \mathbb{R}^3 \) is a semicone \( uv = z^2, u \geq 0, v \geq 0 \), where \( u, v, z \) are coordinates in \( \mathbb{R}^3 \). Obviously \( \mathcal{E}_{Z_2}(2) = \rho^*\mathcal{E}(3) \cong \mathcal{E}(3)/m(\rho, 3) \). Transformation \( f \) is a \( \rho \eta \)-germ and \( \rho \circ f(x, y) = (x^2, (x^3 + y^3)^2, x^4 + y^2) \) is a \( Z_2 \)-invariant mapping. A mapping \( F: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) as in (1), i.e. such that \( \rho \circ f = F \circ \rho \), could be defined by \((u, v, z) \mapsto (u, u^3 + 2z^3 + v^3, u^2 + vz)\). Let us consider the ideal \( I \overset{\text{def}}{=} \langle u, z^2, zv, v^3 \rangle_{\mathcal{E}(3)} \). By straightforward checking we get

\[
I = \langle u, u^3 + 2z^3 + v^3, u^2 + vz, uv - z^2 \rangle_{\mathcal{E}(3)} \subseteq (F^*m_3(3) \cdot \mathcal{E}(3) + m(\rho, 3) \cdot \mathcal{E}(3)).
\]

It is easy to observe that \( m^3(3) \subseteq I + m^4(3) \), so \( m^3(3) \subseteq I \) by Nakayama’s lemma [1]. Now a simple computation shows that \( 1, z, v, v^2 \) represent generators of
the real vector space $\mathcal{E}(3)/I$ and so they generate $\mathcal{E}(3)/(F^*m(3) \cdot \mathcal{E}(3) + m(\rho, 3))$, the real vector space. By Corollary 1 (for $A^* = \mathcal{E}(3)$ and $A = \mathcal{E}(3)/m(\rho, 3)$) they represent generators of module $A$ over $f^*\mathcal{E}_{Z_2}(2) \cong \mathcal{E}_{\eta}(2)$. Now considering an $f^*\mathcal{E}_{Z_2}(2)$-module isomorphism $\rho^* : A \rightarrow \mathcal{E}_{Z_2}(2)$ we find that their combinations with $\rho$, i.e. $1, xy, y^2, y^4$ generate $\mathcal{E}_{Z_2}(2)$ over $f^*\mathcal{E}_{Z_2}(2)$.

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