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A COMPACT SEMILATTICE ON THE HILBERT CUBE WITH NO INTERVAL HOMOMORPHISM

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ABSTRACT. An example for a compact semilattice with no interval homomorphism is given. The underlying space of the semilattice is homeomorphic to the Hilbert cube.

Introduction. A compact semilattice S is a compact, idempotent, commutative topological semigroup with identity. The standard example of such a compact semilattice is the unit interval $[0,1]$ under the operation $xy = \max\{x, y\}$. It is an old problem to decide whether a compact semilattice allows an embedding into a power of $[0,1]$, i.e. whether there are enough continuous semilattice homomorphisms into the unit interval to separate points. Jimmie D. Lawson [**4**] constructed the first two examples of compact semilattices with no nontrivial continuous homomorphisms into $[0,1]$. Up to today, only one other such example is known, and this example is derived from a compact convex set without extreme points. The constructions for those examples are fairly complicated.

Thus it became an interesting question to find topological restrictions which force a compact semilattice to allow enough semilattice homomorphisms into the unit interval to separate the points. Again Jimmie Lawson [**3**] has shown that a topological semilattice on a finite dimensional Peano continuum or on a compact totally disconnected space must have this property. His two counterexamples live on a one-dimensional metric continuum and on an infinite dimensional Peano continuum, respectively.

In this note, I shall give an easy construction of a compact semilattice L without any nontrivial homomorphisms into the unit interval. As a topological space, L will be homeomorphic to the Hilbert cube. Furthermore, L has the interesting property that is order generated by its irreducible elements.

The construction. Let $x = (x_n)_n$ and $y = (y_n)_n \in [0,1]^{\mathbf{N}}$ be two sequences. We define a new sequence $x \vee y$ by $(x \vee y)_n = \max\{x_n, y_n\}$. Under the operation $\vee$, the set $[0,1]^{\mathbf{N}}$ is a compact semilattice. For every positive integer n define a mapping $f_n\colon [0,1]^{\mathbf{N}} \to [0,1]$ by

$$f_n(x_1, x_2, x_3, \dots) = \ln\left(1 + \sum_{i=1}^{n} x_i\right) / \ln(1+n).$$

Clearly, each f_n is monotone increasing and continuous. Furthermore, we have

(1) $$f_n(x \vee y) \leq \max\{f_n(x), f_n(y)\} + \ln 2/\ln(n+1),$$

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i.e. the mappings f_n are almost semilattice homomorphisms. Indeed, assume that $x_1 + \cdots + x_n \le y_1 + \cdots + y_n$. Then

$$\begin{aligned}\ln(1 + \max\{x_1, y_1\} + \cdots + \max\{x_n, y_n\})\\ \le \ln(2 + (x_1 + y_2) + \cdots + (x_n + y_n))\\ \le \ln(2[1 + y_1 + \cdots + y_n])\\ = \max\{\ln(1 + x_1 + \cdots + x_n), \ln(1 + y_1 + \cdots + y_n)\} + \ln 2.\end{aligned}$$

Define

$$f\colon [0,1]^{\mathbf{N}} \to [0,1], \qquad z \mapsto \sup_{n\in N} f_n(x).$$

It follows from [**1**, VI.4.1, VI.4.2] that the set

$$S = \{(x, r) \in [0,1]^{\mathbf{N}} \times [0,1] \colon f(x) \le r\}$$

is a closed subset of $[0,1]^{\mathbf{N}} \times [0,1]$ and a compact semilattice under the operation $(x, r) \cdot (y, s) = (x \vee y, \max\{r, s, f(x \vee y)\})$. Let $z \in [0,1]^{\mathbf{N}}$ be the element with all coordinates 0, and let $\phi\colon S \to [0,1]$ be an arbitrary continuous semilattice homomorphism. We show:

$$\phi(z, 0) = \phi(z, 1). \tag{2}$$

Suppose not. Then $\phi(x, 0) < \phi(z, 1)$; w.l.o.g. we may assume that $\phi(z, 0) = 0$ and $\phi(z, 1) = 1$. The set $U = \{x, s) \in S \colon \phi(x, s) < 1/2\}$ would be an open neighborhood of $(z, 0)$ closed under the semilattice operation and not containing $(z, 1)$ in its closure. Let $e_n \in [0,1]^{\mathbf{N}}$ be the nth unit vector. Then $\alpha_n = (e_n, \ln 2/\ln(n+1)) \in S$. Since $\lim_{n\to\infty} \alpha_n = (z, 0)$ there is an integer N such that $\alpha_n \in U$ for all $n \ge 0$. Let $u_n \in [0,1]^{\mathbf{N}}$ be the vector having 0's in the first $n - 1$ coordinates and 1's in all the others. Clearly $f(u_n) = 1$ for all n. Since $(u_m, 1) = \lim_{k\to\infty} \alpha_m \cdots \alpha_{m+k}$, we conclude that $(u_m, 1) \in \overline{U}$ for every $m > N$, and since $\lim_{m\to\infty}(u_m, 1) = (z, 1)$, we arrive at the contradiction $(z, 1) \in \overline{U}$.

Consider the (closed) equivalence relation Θ on S which has $I = \{(x, 1) \in S \colon x \in [0,1]^{\mathbf{N}}\}$ as its only nontrivial equivalence class. Then I is an ideal of the semigroup S, and the Rees quotient $S/I = S/\Theta$ carries again in canonical way the structure of a compact semilattice. Now (2) implies that for every continuous semilattice homomorphism $\psi\colon S/I \to [0,1]$ we have $\psi(I) = \psi\{(z, 0)\}$. Since in S/I the elements I and $\{(z, 0)\}$ act as zero element and identity, respectively, it follows that every continuous semilattice homomorphism $\phi\colon S/I \to [0,1]$ is constant.

It remains to show that as a topological space, S/I is homeomorphic to the Hilbert cube. Firstly, we verify that S is homeomorphic to the Hilbert cube. For every positive integer N, let

$$\rho_n\colon [-n\ln 2, \ln(1 + 1/2^n)] \to [0,1], \qquad x \mapsto e^x - 1/2^n.$$

Clearly, each ρ_n is a homeomorphism, and therefore the map

$$\rho\colon \prod_{1\le n} [-n\ln 2, \ln(1 + 1/2^n)] \to [0,1]^{\mathbf{N}}, \qquad (x_n)_n \mapsto (\rho_n(x_n))_n,$$

is also a homeomorphism. For each n we have

$$f_n \circ \rho(x_1, x_2, \dots) = \ln(1/2^{n+1} + e^{x_1} + \cdots + e^{x_n})/\ln(1+n),$$

and it follows from the Minkowski inequality that this is a convex function. Hence $(\rho \times \mathrm{id})^{-1}(S)$ is a convex set, i.e. S is homeomorphic to an infinite dimensional compact convex subset of Hilbert space. It follows now from a classical result of O. H. Keller [**2**] that S is homeomorphic to the Hilbert cube. Finally, we obtain that $(\rho \times \mathrm{id})^{-1}(I)$ is a face of the compact convex set $(\rho \times \mathrm{id})^{-1}(S)$. It now follows easily from Hilbert cube theory that S/I is homeomorphic to the Hilbert cube.

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