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ON A ZETA FUNCTION ASSOCIATED TO TERNARY ZERO FORMS

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ABSTRACT. We rederive a relation due to Eie between a zeta function for ternary quadratic forms and the Riemann zeta function, correcting a factor corresponding to the prime $p = 2$.

1. Introduction. A recent paper by Eie [**1**] obtained a relationship between a zeta function associated to ternary quadratic forms and the ordinary Riemann zeta function. However, some technical errors in the derivation resulted in an incorrect and awkward factor corresponding to the prime $p = 2$. In view of potential applications of the result to dimension formulas for spaces of Siegel modular forms, it seems worthwhile to establish the correct formula, which we do in Theorem A. The $p = 2$ factor appears in a simpler, more elegant form.

2. Definitions and statement of Theorem A. We follow the notation in [**1**]. Let s_{13} and s_2 be nonzero integers and define

$$\Delta(s_{13}, s_2) = \left\{ S = \begin{bmatrix} 0 & 0 & s_{13} \\ 0 & s_2 & s_{23} \\ s_{13} & s_{23} & s_3 \end{bmatrix} \mid s_{23}, s_3 \in Z \right\}.$$

(Observe that every ternary zero form is equivalent to one given by such a matrix.) Let

$$P = \left\{ U = \begin{bmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} \mid u, v, w \in Z \right\}.$$

Then P acts on $\Delta(s_{13}, s_2)$ by $S \to {}^tUSU$. Let $\mu(s_{13}, s_2)$ be the number of distinct "orbits" formed under the action of P: $\mu(s_{13}, s_2) = |\Delta(s_{13}, s_2)/P|$. Define the zeta function

$$\xi(t) = \sum_{s_2 \neq 0} \sum_{s_{13}=1}^{\infty} \frac{\mu(s_{13}, s_2)}{|s_2 s_{13}^2|^t}.$$

(Note that $|s_2 s_{13}^2| = |\det S|$.) We shall prove

THEOREM A. *For* $\operatorname{Re}(t) \geq 1$, *we have*

$$\xi(t) = 2\left[2 - \frac{(1-2^{-t})(1-2^{1-2t})(1-2^{1-3t})}{(1-2^{-3t})}\right] \frac{\varsigma(t)\varsigma(2t-1)\varsigma(3t-1)}{\varsigma(3t)}, \tag{1}$$

where $\varsigma(t)$ *is the Riemann zeta function.*

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3. Proof of Theorem A. We begin by calculating $\mu(s_{13}, s_2)$.

PROPOSITION. *Let $s_{13} = ad$ and $s_2 = bd$ with $d = (s_{13}, s_2)$. Let $d = 2^r D$ with D odd. Then*

$$(2)\qquad \mu(s_{13}, s_2) = \begin{cases} 2a\sum_{h=1}^{d}(h,d) & \text{if } 2 \mid ab, \\ a\sum_{h=1}^{d}(2h,d) & \text{if } 2 \nmid ab \text{ and } r = 0, \\ a\sum_{h=1}^{d}(2h,d) + 2^r a\sum_{h=1}^{D}(h,D) & \text{if } 2 \nmid ab \text{ and } r \geq 1. \end{cases}$$

PROOF. It is shown in [**1**] that $S \equiv S' \bmod(P)$ if and only if $s'_{23} \equiv s_{23} \bmod(d)$ and $s'_3 \equiv s_3 \bmod(l(s_{23}))$, where $l(s_{23})$ is the (positive) generator of the principal ideal

$$\begin{aligned} G &= \{2ks_{23}a + k^2a^2bd + 2nad \mid k, n \in Z\} \\ &= \{a(2s_{23}k + d(abk^2 + 2n)) \mid k, n \in Z\}. \end{aligned}$$

Thus $\mu(s_{13}, s_2) = \sum_{s_{23}=1}^{d} l(s_{23})$, so it suffices to compute $l(s_{23})$. If $2|ab$, then

$$G = \left\{2a\left(s_{23}k + d\left(\frac{ab}{2}k^2 + n\right)\right) \mid k, n \in Z\right\} = \{2a(s_{23}k + dn') \mid k, n' \in Z\},$$

so $l(s_{23}) = 2a(s_{23}, d)$, giving the result in that case. If $2 \nmid ab$, then let $2s_{23} = \sigma'\delta$ and $d = d'\delta$ with $\delta = (2s_{23}, d)$. We have $G = \{a\delta(\sigma'k + d'(abk^2 + 2n)) \mid k, n \in Z\}$. It is easy to see that the smallest positive value of $\sigma'k + d'(abk^2 + 2n)$ is 1 if $2 \mid \sigma'd'$ and 2 if $2 \nmid \sigma'd'$. We can thus express $\mu(s_{13}, s_2)$ as

$$\mu(s_{13}, s_2) = a\sum_{s_{23}=1}^{d}(2s_{23}, d) + a\sum_{s_{23}=1, 2\nmid\sigma'd'}^{d}(2s_{23}, d) \quad \text{if } 2 \nmid ab.$$

Now let $d = 2^r D$ with D odd. If $r = 0$, then $2 \nmid d \Rightarrow 2 \mid \sigma'$ for all s_{23}, so that the second sum is vacant, which proves the middle formula of the proposition. If $r \geq 1$, then $2 \nmid \sigma'd' \Rightarrow 2 \nmid d' \Rightarrow 2^r \mid \delta \Rightarrow 2^r \mid 2\sigma \Rightarrow 2s_{23} = 2^r S \Rightarrow s_{23} = 2^{r-1}S$, and thus

$$\sum_{s_{23}=1, 2\nmid\sigma'd'}^{d}(2s_{23}, d) = \sum_{S=1, 2\nmid S}^{2D}(2^r S, 2^r D) = 2^r\sum_{S=1}^{D}(S, D)$$

since D is odd. This completes the proof of the proposition.

The proof of Theorem A is now a relatively straightforward computation. We shall only outline the steps.

$$\begin{aligned} (3)\qquad \xi(t) &= \sum_{s_2\neq 0}\sum_{s_{13}=1}^{\infty}\frac{\mu(s_{13}, s_2)}{|s_2 s_{13}^2|^t} \\ &= 2\sum_{(a,b)=1}\sum_{d=1}^{\infty}\frac{\mu(ad, bd)}{a^{2t}b^t d^{3t}} \qquad (a, b > 0) \\ &= 4\left[\sum_{(a,b)=1, 2|ab}\frac{1}{a^{2t-1}b^t}\right]\left[\sum_{d=1}^{\infty}\frac{c_1(d)}{d^{3t}}\right] \\ &\quad + 2\left[\sum_{(a,b)=1, 2\nmid ab}\frac{1}{a^{2t-1}b^t}\right]\left[\sum_{d=1}^{\infty}\frac{c_2(d)}{d^{3t}} + \sum_{d=1, 2|d}^{\infty}\frac{c'_1(d)}{d^{3t}}\right] \end{aligned}$$

where $c_1(d) = \sum_{h=1}^{d}(h,d)$, $c_2(d) = \sum_{h=1}^{d}(2h,d)$, and $c_1'(d) = 2^r c_1(D)$ with D odd and $d = 2^r D$. The sums over a and b are easily converted into zeta functions; in general one has

$$\text{(4)} \qquad \sum_{(a,b)=1,2|ab} \frac{1}{a^s b^t} = \left(1 - \frac{1}{2^{s+t}}\right)^{-1} \left(\frac{1}{2^t} + \frac{1}{2^s} - \frac{1}{2^{s+t-1}}\right) \frac{\varsigma(\xi)\varsigma(t)}{\varsigma(s+t)},$$

$$\text{(5)} \qquad \sum_{(a,b)=1,2\nmid ab} \frac{1}{a^s b^t} = \left(1 - \frac{1}{2^{s+t}}\right)^{-1} \left(1 - \frac{1}{2^t}\right) \left(1 - \frac{1}{2^s}\right) \frac{\varsigma(s)\varsigma(t)}{\varsigma(s+t)}.$$

Finally, we have the three identities

$$\text{(6)} \qquad \sum_{d=1}^{\infty} \frac{c_1(d)}{d^{3t}} = \frac{\varsigma(3t-1)^2}{\varsigma(3t)},$$

$$\text{(7)} \qquad \sum_{d=1}^{\infty} \frac{c_2(d)}{d^{3t}} = \left(1 - \frac{1}{2^{3t}}\right)^{-1} \frac{\varsigma(3t-1)^2}{\varsigma(3t)},$$

$$\text{(8)} \qquad \sum_{d=1,2|d}^{\infty} \frac{c_1'(d)}{d^{3t}} = \frac{1}{2^{3t-1}} \left(1 - \frac{1}{2^{3t-1}}\right) \left(1 - \frac{1}{2^{3t}}\right)^{-1} \frac{\varsigma(3t-1)^2}{\varsigma(3t)}.$$

(The first identity is proved by showing that $c_1(d)$ is multiplicative and $c_1(p^m) = (m+1)p^m - mp^{m-1}$; the other two identities follows from the first.) When (4)–(8) are plugged into (3), the result is (1).

REMARK. Although the $p = 2$ factor in (1) is different from that given in [**1**], the value at $t = 2$ is unchanged: $\xi(2) = 65\varsigma(2)\xi(3)\varsigma(5)/24\varsigma(6)$.

REFERENCES

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