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MORRIS NEWMAN

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ABSTRACT. Let R be a principal ideal ring and $M_{k,n}$ the set of $k \times n$ matrices over R. The following statments are proved: (a) If $k \leq n/3$ then any primitive element of $M_{k,n}$ occurs as the first k rows of the commutator of two elements of $\mathrm{SL}(n, R)$. (b) If every element of $\mathrm{SL}(3, R)$ is the product of at most c_3 commutators, then every element of $\mathrm{SL}(n, R)$ is the product of at most c_n commutators, where $c_n < c \log n + c_3 - 3$, $c = 2\log(3/2) = 4.932\ldots$, and $n \geq 3$. (c) If $n \geq 3$, then every element of $\mathrm{SL}(n, Z)$ is the product of at most $c \log n + 40$ commutators, where c is given in (b) above

1. Introduction. Let R denote an arbitrary principal ideal ring, and let $M_{k,n} = M_{k,n}(R)$ denote the set of $k \times n$ matrices over R, $M_n = M_n(R)$ the set of $n \times n$ matrices over R. If $1 \leq k \leq n$, an element α of $M_{k,n}$ is said to be *primitive* if $d_k(\alpha) = 1$, where $d_k(\alpha)$ is the kth determinantal divisor of α. As usual, let $\Gamma = \mathrm{SL}(n, R)$ denote the multiplicative group of matrices of M_n of determinant 1. The primitive elements of $M_{k,n}$ for $1 \leq k < n$ are just those which may be completed to an element of Γ.

It is well known that in many cases Γ coincides with its commutator subgroup Γ'. This happens, for example, when R is a euclidean ring of characteristic not 2, and $n \geq 3$. Assume from now on that $\Gamma = \Gamma'$. The question considered here is whether or not every element of Γ may be represented as the product of a bounded number of commutators, where the bound depends only on n. Accordingly, when Γ has this property, we define $c_n = c_n(R)$ as the least positive integer such that every element of Γ is the product of at most c_n commutators. In order to complete the definition, we define c_n to be infinity, when Γ does not have this property.

In this paper we reduce the question to the case $n = 3$, and prove that the answer is in the affirmative for $R = Z$ and $n \geq 3$. The result in this case depends on a crucial theorem of D. Carter and G. Keller (see [**1**]) who prove that if A is any matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$$

belonging to $\mathrm{SL}(3, Z)$, then A is the product of at most 41 elementary matrices. (An elementary matrix of Γ is any matrix of the form $I + xE_{ij}$, where $I = I_n$ is the $n \times n$ identity matrix, i, j are distinct integers such that $1 \leq i, j \leq n$, E_{ij} is the $n \times n$ matrix with a 1 in position (i, j) and 0 elsewhere, and x is any element of R.) Since every elementary matrix is also a commutator, the result cited above

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implies that any matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$$

belonging to $\mathrm{SL}(3, Z)$ is the product of at most 41 commutators.

The bound we obtain is surprisingly small—in fact of order $\log n$— and prompts the conjecture that every element of $\mathrm{SL}(n, Z)$ is the product of at most c commutators for some absolute constant c, provided that $n \geq 3$. The case $n = 2$ is an exception, since then $(\Gamma\colon \Gamma') = 12$.

2. Statement of results. We prove three theorems. The first is a completion theorem which generalizes Theorem 6 of [**2**], and the others are concerned with bounds for c_n.

THEOREM 1. *Let α be a primitive element of $M_{k,n}$, where $k \leq n/3$. Then matrices A, B of* $\mathrm{SL}(n, R)$ *exist such that α coincides with the first k rows of* $[A, B] = ABA^{-1}B^{-1}$.

THEOREM 2. *Suppose that $n \geq 3$. Then*

$$c_n(R) < c \log n + c_3(R) - 3, \tag{1}$$

where

$$c = 2/\log(3/2) = 4.932\ldots. \tag{2}$$

THEOREM 3. *Suppose that $n \geq 3$. Then every element of* $\mathrm{SL}(n, Z)$ *is the product of at most $c \log n + 40$ commutators, where c is given by* (2).

2. Proofs.

PROOF OF THEOREM 1. Write $\alpha = [U\,V]$, where $U \in M_k$ and $V \in M_{k,n-k}$. Let S be any element of $\mathrm{SL}(n-k, R)$. We have

$$\begin{bmatrix} I_k & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} U & V \\ & * \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} U & VS \\ & * \end{bmatrix}.$$

Thus it is sufficient to prove the theorem for $[U\,VS]$, since conjugates of commutators are commutators. Choose S so that $VS = [W\,0]$, where $W \in M_k$ and 0 is a $k \times (n-2k)$ block of zeros. Since $k \leq n/3$, it is sufficient to prove the theorem for $[U\,W\,0]$, where now 0 is a $k \times k$ block of zeros.

Since $[U\,W]$ is a primitive element of $M_{k,2k}$, matrices X, Y exist which belong to M_k such that

$$\begin{bmatrix} U & W \\ X & Y \end{bmatrix} \in \mathrm{SL}(2k, R).$$

Put

$$\begin{bmatrix} U & W \\ X & Y \end{bmatrix}^{-1} = \begin{bmatrix} U_1 & W_1 \\ X_1 & Y_1 \end{bmatrix}.$$

We note that if I stands for the $k \times k$ identity matrix, then

$$A = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \in \mathrm{SL}(3, R),$$

$$B = \begin{bmatrix} U_1 & W_1 & 0 \\ X_1 & Y_1 & 0 \\ 0 & 0 & I \end{bmatrix} \in \mathrm{SL}(3k, R).$$

We now have

$$\begin{aligned} [A, B] &= ABA^{-1}B^{-1} \\ &= \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} U_1 & W_1 & 0 \\ X_1 & Y_1 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} U & W & 0 \\ X & Y & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & I \\ U_1 & W_1 & 0 \\ X_1 & Y_1 & 0 \end{bmatrix} \begin{bmatrix} X & Y & 0 \\ 0 & 0 & I \\ U & W & 0 \end{bmatrix} = \begin{bmatrix} U & W & 0 \\ * & * & * \\ * & * & * \end{bmatrix}. \end{aligned}$$

This completes the proof.

For the remaining theorems, two lemmas are required.

LEMMA 1. *Suppose that $n \geq 2$. Then a matrix C exists such that both C and $C - I$ belong to* $\mathrm{SL}(n, R)$.

PROOF. Let $q(x)$ be any monic polynomial of degree $n - 2$ belonging to $R[x]$, and let $p(x) = x(x-1)q(x) + (-1)^n$. Let C be the companion matrix of $p(x)$. Since $\det(C) = (-1)^n p(0) = 1$, C belongs to $\mathrm{SL}(n, R)$. The characteristic polynomial of $C - I$ is $p(x+1)$; and since $\det(C - I) = (-1)^n p(1) = 1$, $C - I$ also belongs to $\mathrm{SL}(n, R)$. This completes the proof.

LEMMA 2. *Suppose that $n \geq 3$, and let*

$$E = \begin{bmatrix} I_r & 0 \\ A & I_s \end{bmatrix} \in \mathrm{SL}(n, R),$$

where A is any element of $M_{s,r}$. Then E is a commutator.

PROOF. Either r or s must be ≥ 2; we may assume that $s \geq 2$. Let $C \in \mathrm{SL}(s, R)$ be such that $C - I$ also belongs to $\mathrm{SL}(s, R)$ (Lemma 1). Let B be an element of $M_{s,r}$ to be determined later. We have

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ B & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ B & C \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & C^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ (I - C)B & I \end{bmatrix}. \end{aligned}$$

Since $I - C$ is unimodular, we may choose $B = (I - C)^{-1} A$. For this B, E becomes a commutator, and the proof is concluded.

PROOF OF THEOREM 2. Let L be any matrix of $\mathrm{SL}(n, R)$. Put $k = [n/3]$. By Theorem 1, there is a matrix K of $\mathrm{SL}(n, R)$ such that the first k rows of K coincide with the first k rows of L^{-1}, and K is a commutator. Then

$$KL = \begin{bmatrix} I_k & 0 \\ A & L_1 \end{bmatrix},$$

where $A \in M_{n-k,k}$ and $L_1 \in \mathrm{SL}(n-k, R)$. Thus

$$L = K^{-1}\begin{bmatrix} I_k & 0 \\ A & L_1 \end{bmatrix} = K^{-1}\begin{bmatrix} I_k & 0 \\ A & I_{n-k} \end{bmatrix}\begin{bmatrix} I_k & 0 \\ 0 & L_1 \end{bmatrix}.$$

By Lemma 2, the matrix

$$\begin{bmatrix} I_k & 0 \\ A & I_{n-k} \end{bmatrix}$$

is a commutator, and so L has been expressed as the product of two commutators and the matrix $\left[\begin{smallmatrix} I_K & 0 \\ 0 & L_1 \end{smallmatrix}\right]$ where the order of L_1 is $n - [n/3]$.

We now repeat this process, obtaining the sequence of integers

$$\begin{aligned} n_0 &= n, \\ n_1 &= n_0 - [n_0/3], \\ &\dots \\ n_{k+1} &= n_k - [n_k/3], \\ &\dots \end{aligned}$$

Since $n_k = 3[n_k/3] + r_k$, where $r_k = 0, 1, 2$, we have the inequalities

$$\tfrac{2}{3}n_k \leq n_{k+1} \leq \tfrac{2}{3}n_k + \tfrac{2}{3}, \qquad k = 0, 1, \dots,$$

which readily imply that

$$\left(\tfrac{2}{3}\right)^k n \leq n_k < \left(\tfrac{2}{3}\right)^k n + 2, \qquad k = 0, 1, \dots.$$

We now choose k so that $3 \leq \left(\tfrac{2}{3}\right)^k n$, $\left(\tfrac{2}{3}\right)^k n + 2 \leq 7$, which is equivalent to

$$\begin{gathered} \frac{n}{5} \leq \left(\frac{3}{2}\right)^k \leq \frac{n}{3}, \\ \frac{\log(n/5)}{\log(3/2)} \leq k \leq \frac{\log(n/3)}{\log(3/2)}. \end{gathered}$$

Since

$$\frac{\log(n/3)}{\log(3/2)} - \frac{\log(n/5)}{\log(3/2)} = \frac{\log(5/3)}{\log(3/2)} > 1,$$

such a k exists. It then follows that L is the product of $2k - 2$ commutators, and a matrix of the form $\left[\begin{smallmatrix} I_{n-r} & 0 \\ 0 & T \end{smallmatrix}\right]$, where $T \in \mathrm{SL}(r, Z)$ and $r = 3, 4, 5, 6$. Employing this reduction again for $r = 4, 5, 6$ we find that T is the product of at most $c_3 + s$ commutators, where $s = 0$ for $r = 3$, $s = 2$ for $r = 4$, $s = 4$ for $r = 5$, and $s = 4$ for $r = 6$. It follows that in all cases, an upper bound for c_n is given by $2k - 2 + c_3 + 4 = 2k + c_3 + 2$. Since $k \leq \log(n/3)/\log(3/2)$, we have that

$$\begin{aligned} c_n &\leq \frac{2\log n}{\log(3/2)} + c_3 + 2 - \frac{2\log 3}{\log(3/2)} \\ &< \frac{2\log n}{\log(3/2)} + c_3 - 3, \end{aligned}$$

the desired inequality. This completes the proof.

PROOF OF THEOREM 3. Theorem 3 follows directly from Theorem 2 and the result of D. Carter and G. Keller cited above, since $c_3(Z) \leq 43$.

4. Some open questions. A number of interesting open problems are suggested by these results:

(a) Suppose that G is a finitely presented perfect group. When is it possible to deduce from the presentation of G alone that an absolute constant c exists such that every element of G is the product of at most c commutators? This property of a group is of importance for classification purposes, and the results of this paper should be of use whenever G has a faithful representation as a subgroup of $\mathrm{SL}(n, Z)$ for some n.

(b) The commutator subgroup $\mathrm{SL}'(2, Z)$ of $\mathrm{SL}(2, Z)$ is of index 12, and is a free group of rank 2. Somewhat more generally, let $G = \{x, y\}$ be a free group of rank 2, freely generated by x and y. Then if k is a positive integer, the element $z = (xyx^{-1}y^{-1})^k$ is the product of k commutators. Prove or disprove that z is not the product of fewer than k commutators.

(c) Determine the perfect subgroups of finite index of $\mathrm{SL}(n, Z)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106