EXACT SEQUENCES FOR
GENERALIZED TOEPLITZ OPERATORS

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ABSTRACT. Let $\mathcal{T}$ be the $C^*$-algebra generated by the Toeplitz operators on $H^2$ of the unit circle, and let $C$ be the $\mathcal{T}$-ideal generated by \{${T_\varphi T_\psi - T_{\varphi \psi}} : \varphi, \psi \in L^\infty$\}. It is well known that $\mathcal{T}/C$ is naturally $*$-isomorphic to $L^\infty$. Several authors have obtained a similar result for other classes of Toeplitz operators. In the present paper a general theorem is proved which establishes the relevant isomorphism for a wide class of generalized Toeplitz operators.

1. Introduction. In the study of Toeplitz operators $T_\varphi$ on the Hardy space $H^2$ on the unit circle, two objects of interest are the Toeplitz algebra $\mathcal{T}$, which is the $C^*$-operator algebra generated by the Toeplitz operators, and its semicommutator ideal $C$, which is the $\mathcal{T}$-ideal generated by \{${T_\varphi T_\psi - T_{\varphi \psi}} : \varphi, \psi \in L^\infty$\}. A basic fact concerning these objects is that $\mathcal{T}/C$ is naturally isomorphic to $L^\infty$, and a related fact is the spectral inclusion theorem: $\mathcal{R}(\varphi) \subseteq \sigma(T_\varphi)$, where $\mathcal{R}(\varphi)$ denotes the essential range of $\varphi$. For proofs of these facts and some consequences, see Chapter 7 of [3].

Recently there has been interest in the study of operators similar to the classical Toeplitz operators which act on Hilbert spaces of analytic functions other than $H^2$. For many of these operators the isomorphism $\mathcal{T}/C$ with some naturally related function algebra is proven and a related spectral inclusion theorem is shown; see for example [1, 2, and 6]. It is the purpose of this note to show how an idea of Davie and Jewell in [2] can be used to prove an isomorphism theorem for a very general class of operators. This result in fact shows that in all cases, the isomorphism theorem and the spectral inclusion theorem are equivalent.

2. The main result. Let $\mu$ be a positive measure on some measure space, $H$ a closed subspace of $L^2(\mu)$, and $P$ the orthogonal projection of $L^2(\mu)$ onto $H$. To $\varphi \in L^\infty(\mu)$ we associate the generalized Toeplitz operator $T_\varphi : H \rightarrow H$ defined by

$$T_\varphi f = P(\varphi f) \quad \text{for } f \in H.$$  

Thus $T_\varphi = PM_\varphi|_H$, where $M_\varphi$ is the operation of multiplication by $\varphi$.

Now let $A$ be a $C^*$-subalgebra of $L^\infty(\mu)$, with maximal space $M$. For notational convenience, if $\varphi \in A$ and $x \in M$ we will write $\varphi(x)$ for the action of $x$ on $\varphi$. Let $\mathcal{T}$ be the $C^*$-operator algebra generated by \{${T_\varphi} : \varphi \in A$\}, and let $C$ be the $\mathcal{T}$-ideal generated by \{${T_\varphi T_\psi - T_{\varphi \psi}} : \varphi, \psi \in L^\infty$\}. It is well known that $\mathcal{T}/C$ is naturally $*$-isomorphic to $L^\infty$. Several authors have obtained a similar result for other classes of Toeplitz operators. In the present paper a general theorem is proved which establishes the relevant isomorphism for a wide class of generalized Toeplitz operators.
generated by \( \{ T_\varphi T_\psi - T_{\varphi \psi} : \varphi, \psi \in A \} \). We then have

**Theorem.** \( \mathcal{T}/\mathcal{C} \) is naturally \(*\)-isometrically isomorphic to \( C(M_1) \), where

\[
M_1 = \{ x \in M | \varphi \in A, \varphi(x) = 0 \Rightarrow M_\varphi \text{ is not bounded below on } H \}.
\]

That is, the sequence

\[
0 \to I \to A \xrightarrow{\Phi} \mathcal{T}/\mathcal{C} \to 0
\]

is exact, where \( \Phi(\varphi) = T_\varphi + \mathcal{C} \) and \( I = \{ \varphi \in A | \varphi \equiv 0 \text{ on } M_1 \} \).

**Proof.** We first observe that \( \varphi \mapsto T_\varphi \) is linear and satisfies

(a) \( T_\varphi = I \),

(b) \( \| T_\varphi \| \leq \| M_\varphi H \| \leq \| \varphi \|_\infty \),

(c) \( T_\varphi^* = T_\varphi \),

(d) \( \| T_\varphi f \|_2^2 \leq \| M_\varphi f \|_2^2 \leq \| \varphi \|_\infty (\| T_\varphi |f| f \|) \) for \( f \in H \),

(e) \( \varphi \geq 0 \Rightarrow T_\varphi \geq 0 \).

Only the second inequality in (d) requires any comment: if \( f \in H \), then

\[
\| M_\varphi f \|_2^2 = \int |\varphi f|^2 \, d\mu \leq \|\varphi\|_\infty \int |\varphi|^2 \, d\mu = \|\varphi\|_\infty \| (\varphi |f| f) \| = \|\varphi\|_\infty \| (T_\varphi |f| f) \|.
\]

It is an easy consequence of (b) that \( M_1 \) is closed. By the definition of \( \mathcal{C} \), \( \Phi \) is an algebra homomorphism. Define \( M_2 \) to be the zero set of the ideal \( \ker \Phi \subset A \). The content of the Theorem is that \( M_2 = M_1 \).

To show that \( M_1 \subseteq M_2 \) we use a direct adaptation of the proof of Theorem 2.2 in [2]. Let \( x_0 \in M_1 \) and let \( \varphi \in A \) be such that \( \varphi(x_0) = 1 \). We wish to show that \( T_\varphi \notin \mathcal{C} \); to this end, let

\[
S = \sum_j \left( \prod_i T_{\xi_j^{(i)}} \right) (T_{\eta_i} T_{\psi_j} - T_{\eta_j \psi_j}) \left( \prod_i T_{k_j^{(i)}} \right)
\]

be a typical generating element of \( \mathcal{C} \). Relabel the function \( \varphi \) as \( \varphi_0 \) and relabel all the functions \( \xi_j^{(i)}, \eta_j, \psi_j, \eta_j \psi_j, k_j^{(i)} \) as \( \varphi_1, \ldots, \varphi_N \). Define \( \Psi = \sum_{k=0}^N |\varphi_j - \varphi_j(x_0)| \). Then \( \Psi(x_0) = 0 \), so since \( x_0 \in M_1 \) there exist \( f_n \in H \) such that \( \| f_n \|_2 = 1 \) and \( \| T_\Psi f_n \|_2 \to 0 \). By (e) \( T_{\varphi_j - \varphi_j(x_0)} \geq 0 \), hence \( \| T_{\varphi_j - \varphi_j(x_0)} f_n \|_2 \to 0 \). By (d) and (a) this implies that \( \| T_{\varphi_j} f_n - \varphi_j(x_0) f_n \|_2 \to 0 \), \( j = 0, \ldots, N \). This shows that \( \| S f_n \|_2 \to 0 \) and \( \| T_\varphi f_n \| \to 0 \), so \( \| T - S \| \geq 1 \). Since \( S \) was a typical generating element of \( \mathcal{C} \), this shows that \( T_\varphi \notin \mathcal{C} \). We have shown that \( \varphi(x_0) = 1 \) implies \( T_\varphi \notin \mathcal{C} \), which shows that \( x_0 \in M_2 \).

To show that \( M_2 \subseteq M_1 \), let \( x \in M \setminus M_1 \). There then exists a \( \varphi \in A, \| \varphi \|_\infty = 1 \), such that \( \varphi(x_0) = 0 \) and \( M_\varphi \) is bounded below on \( H \). By (d), \( T_{|\varphi|} \) is then bounded below. This together with (b) and (e) implies that \( \| I - T_{|\varphi|} \| < 1 \), so \( T_{|\varphi|} \) is invertible in \( \mathcal{T} \). By the definition of \( M_2 \) this implies that \( |\varphi| \) does not vanish anywhere on \( M_2 \). Since \( \varphi(x_0) = 0 \) this shows that \( x_0 \notin M_2 \). □

**Remark.** By (d) and (e), \( T_{|\varphi|} \) is invertible iff \( M_{|\varphi|} \) is bounded below on \( H \). Hence an equivalent definition of \( M_1 \) is that \( M_1 \) is the largest of those subsets \( E \subset M \) such that \( \varphi(E) \subset \sigma(T_\varphi) \) for all \( \varphi \in A \).
3. Some examples. If $H \subset L^2(\mu)$ has the property that whenever $\mu(E) > 0$ there exist $f_n \in H$, $\|f_n\|_2 = 1$ such that $\int_E |f_n|^2 \, d\mu \to 1$, then the Theorem shows that $\mathcal{T}/\mathcal{C} \approx A$, since in this case $\mathcal{M}_1 = \mathcal{M}$. The examples in (i) and (ii) below are of this type; the necessary calculation for (i) is easy and standard, and the calculation for (ii) is done in the proof of Theorem 2.1 of [2].

(i) Take $\mu = d\theta$ on $|z| = 1$, $H = H^2$. Then $\mathcal{T}/\mathcal{C} \approx A$. The standard cases are $A = L^\infty$ and $A = C$, the continuous functions on $|z| = 1$. See Chapter 7 of [3].

(ii) Take $\mu$ to be surface measure on $S$, the boundary of the unit ball in $\mathbb{C}^n$, and $H = H^2(S)$. Then $\mathcal{T}/\mathcal{C} \approx A$. The case $A = L^\infty$ was done by Davie and Jewell [2] and the case $A = C(S)$ was done by Coburn [1].

(iii) Take $\mu$ to be two-dimensional Lebesgue measure on $|z| < 1$ and let $H$ be the Bergman space of analytic $L^2(\mu)$ functions. We consider two possibilities for the algebra $A$.

- $A = L^\infty$: We consider $\mathcal{M}$ to be the space of ultrafilters on the lattice of measurable sets modulo sets of measure zero. Then $\mathcal{T}/\mathcal{C} \approx C(\mathcal{M}_1)$ where $\mathcal{M}_1$ consists of those ultrafilters $\mathcal{E}$ having the property that if $E \in \mathcal{E}$ and $\varepsilon > 0$ there exists a “Carleson square” $S_h = \{z = re^{i\theta} | 1 - h < r < 1, |\theta - \theta_0| < h\}$ such that $\mu(E \cap S_h) > (1 - \varepsilon)S_h$. The necessary calculations follow from a theorem of Hastings [4]; see also [6].

- $A$ equals the $C^*$-algebra generated by $H^\infty$: In this case $\mathcal{M}$ is the maximal ideal space of $H^\infty$ and $\mathcal{M}_1$ is the set of “one-point parts” of $\mathcal{M}$ (see [5]). This fact was established in [6]; the use of the present Theorem allows for a considerable simplification of the proof.

References


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