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EXACT SEQUENCES FOR GENERALIZED TOEPLITZ OPERATORS

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ABSTRACT. Let $\mathcal{T}$ be the C^*-algebra generated by the Toeplitz operators on H^2 of the unit circle, and let C be the $\mathcal{T}$-ideal generated by $\{T_\varphi T_\psi - T_{\varphi\psi}\colon \varphi, \psi \in L^\infty\}$. It is well known that $\mathcal{T}/C$ is naturally $*$-isomorphic to L^∞. Several authors have obtained a similar result for other classes of Toeplitz operators. In the present paper a general theorem is proved which establishes the relevant isomorphism for a wide class of generalized Toeplitz operators.

1. Introduction. In the study of Toeplitz operators T_φ on the Hardy space H^2 on the unit circle, two objects of interest are the Toeplitz algebra $\mathcal{T}$, which is the C^*-operator algebra generated by the Toeplitz operators, and its semicommutator ideal $\mathcal{C}$, which is the $\mathcal{T}$-ideal generated by $\{T_\varphi T_\psi - T_{\varphi\psi}\colon \varphi, \psi \in L^\infty\}$. A basic fact concerning these objects is that $\mathcal{T}/\mathcal{C}$ is naturally isomorphic to L^∞, and a related fact is the *spectral inclusion theorem*: $\mathcal{R}(\varphi) \subset \sigma(T_\varphi)$, where $\mathcal{R}(\varphi)$ denotes the essential range of φ. For proofs of these facts and some consequences, see Chapter 7 of [**3**].

Recently there has been interest in the study of operators similar to the classical Toeplitz operators which act on Hilbert spaces of analytic functions other than H^2. For many of these operators the isomorphism $\mathcal{T}/\mathcal{C}$ with some naturally related function algebra is proven and a related spectral inclusion theorem is shown; see for example [**1**, **2**, and **6**]. It is the purpose of this note to show how an idea of Davie and Jewell in [**2**] can be used to prove an isomorphism theorem for a very general class of operators. This result in fact shows that in all cases, the isomorphism theorem and the spectral inclusion theorem are equivalent.

2. The main result. Let μ be a positive measure on some measure space, H a closed subspace of $L^2(\mu)$, and P the orthogonal projection of $L^2(\mu)$ onto H. To $\varphi \in L^\infty(\mu)$ we associate the *generalized Toeplitz operator* $T_\varphi\colon H \to H$ defined by

$$T_\varphi f = P(\varphi f) \quad \text{for } f \in H.$$

Thus $T_\varphi = PM_\varphi|H$, where M_φ is the operation of multiplication by φ.

Now let A be a C^*-subalgebra of $L^\infty(\mu)$, with maximal space $\mathcal{M}$. For notational convenience, if $\varphi \in A$ and $x \in \mathcal{M}$ we will write $\varphi(x)$ for the action of x on φ. Let $\mathcal{T}$ be the C^*-operator algebra generated by $\{T_\varphi\colon \varphi \in A\}$, and let $\mathcal{C}$ be the $\mathcal{T}$-ideal

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generated by $\{T_\varphi T_\psi - T_{\varphi\psi}\colon \varphi,\psi \in A\}$. We then have

THEOREM. *$\mathcal{T}/\mathcal{C}$ is naturally $*$-isometrically isomorphic to $C(\mathcal{M}_1)$, where*

$$\mathcal{M}_1 = \{x \in \mathcal{M} | \varphi \in A,\ \varphi(x) = 0 \Rightarrow M_\varphi \text{ is not bounded below on } H\}.$$

That is, the sequence

$$0 \to I \to A \xrightarrow{\Phi} \mathcal{T}/\mathcal{C} \to 0$$

is exact, where $\Phi(\varphi) = T_\varphi + \mathcal{C}$ and $I = \{\varphi \in A | \varphi \equiv 0 \text{ on } \mathcal{M}_1\}$.

PROOF. We first observe that $\varphi \mapsto T_\varphi$ is linear and satisfies
(a) $T_1 = I$,
(b) $\|T_\varphi\| \leq \|M_\varphi|H\| \leq \|\varphi\|_\infty$,
(c) $T_\varphi^* = T_{\bar{\varphi}}$,
(d) $\|T_\varphi f\|_2^2 \leq \|M_\varphi f\|^2 \leq \|\varphi\|_\infty \langle T_{|\varphi|} f | f\rangle$ for $f \in H$,
(e) $\varphi \geq 0 \Rightarrow T_\varphi \geq 0$.
Only the second inequality in (d) requires any comment: if $f \in H$, then

$$\begin{aligned}\|M_\varphi f\|_2^2 &= \int |\varphi f|^2\, d\mu \leq \|\varphi\|_\infty \int |\varphi|\, |f|^2\, d\mu \\ &= \|\varphi\|_\infty \langle |\varphi| f | f\rangle = \|\varphi\|_\infty \langle T_{|\varphi|} f | f\rangle.\end{aligned}$$

It is an easy consequence of (b) that $\mathcal{M}_1$ is closed. By the definition of $\mathcal{C}$, Φ is an algebra homomorphism. Define $\mathcal{M}_2$ to be the zero set of the ideal $\ker \Phi \subset A$. The content of the Theorem is that $\mathcal{M}_2 = \mathcal{M}_1$.

To show that $\mathcal{M}_1 \subseteq \mathcal{M}_2$ we use a direct adaptation of the proof of Theorem 2.2 in [**2**]. Let $x_0 \in \mathcal{M}_1$ and let $\varphi \in A$ be such that $\varphi(x_0) = 1$. We wish to show that $T_\varphi \notin \mathcal{C}$; to this end, let

$$S = \sum_j \left(\prod_i T_{\xi_j^{(i)}}\right) (T_{\eta_i} T_{\psi_j} - T_{\eta_j \psi_j}) \left(\prod_l T_{k_j^{(l)}}\right)$$

be a typical generating element of $\mathcal{C}$. Relabel the function φ as φ_0 and relabel all the functions $\xi_j^{(i)}, \eta_j, \psi_j, \eta_j\psi_j, k_j^{(l)}$ as $\varphi_1, \ldots, \varphi_N$. Define $\Psi = \sum_{k=0}^N |\varphi_j - \varphi_j(x_0)|$. Then $\Psi(x_0) = 0$, so since $x_0 \in \mathcal{M}_1$ there exist $f_n \in H$ such that $\|f_n\|_2 = 1$ and $\|T_\Psi f_n\|_2 \to 0$. By (e) $T_{|\varphi_j - \varphi_j(x_0)|} \geq 0$, hence $\|T_{|\varphi_j - \varphi_j(x_0)|} f_n\|_2 \to 0$. By (d) and (a) this implies that $\|T_{\varphi_j} f_n - \varphi_j(x_0) f_n\|_2 \to 0$, $j = 0, \ldots, N$. This shows that $\|S f_n\|_2 \to 0$ and $\|T_\varphi f_n\| \to 1$, so $\|T - S\| \geq 1$. Since S was a typical generating element of $\mathcal{C}$, this shows that $T_\varphi \notin \mathcal{C}$. We have shown that $\varphi(x_0) = 1$ implies $T_\varphi \notin \mathcal{C}$, which shows that $x_0 \in \mathcal{M}_2$.

To show that $\mathcal{M}_2 \subseteq \mathcal{M}_1$, let $x \in \mathcal{M} \backslash \mathcal{M}_1$. There then exists a $\varphi \in A$, $\|\varphi\|_\infty = 1$, such that $\varphi(x_0) = 0$ and M_φ is bounded below on H. By (d), $T_{|\varphi|}$ is then bounded below. This together with (b) and (e) implies that $\|I - T_{|\varphi|}\| < 1$, so $T_{|\varphi|}$ is invertible in $\mathcal{T}$. By the definition of $\mathcal{M}_2$ this implies that $|\varphi|$ does not vanish anywhere on $\mathcal{M}_2$. Since $\varphi(x_0) = 0$ this shows that $x_0 \notin \mathcal{M}_2$. □

REMARK. By (d) and (e), $T_{|\varphi|}$ is invertible iff $M_{|\varphi|}$ is bounded below on H. Hence an equivalent definition of $\mathcal{M}_1$ is that $\mathcal{M}_1$ is the largest of those subsets $E \subset \mathcal{M}$ such that $\varphi(E) \subset \sigma(T_\varphi)$ for all $\varphi \in A$.

3. Some examples. If $H \subset L^2(\mu)$ has the property that whenever $\mu(E) > 0$ there exist $f_n \in H$, $\|f_n\|_2 = 1$ such that $\int_E |f_n|^2 \, d\mu \to 1$, then the Theorem shows that $\mathcal{T}/\mathcal{C} \approx A$, since in this case $\mathcal{M}_1 = \mathcal{M}$. The examples in (i) and (ii) below are of this type; the necessary calculation for (i) is easy and standard, and the calculation for (ii) is done in the proof of Theorem 2.1 of [**2**].

(i) Take $\mu = d\theta$ on $[|z| = 1]$, $H = H^2$. Then $\mathcal{T}/\mathcal{C} \approx A$. The standard cases are $A = L^\infty$ and $A = C$, the continuous functions on $[|z| = 1]$. See Chapter 7 of [**3**].

(ii) Take μ to be surface measure on S, the boundary of the unit ball in $\mathbf{C}^n$, and $H = H^2(S)$. Then $\mathcal{T}/\mathcal{C} \approx A$. The case $A = L^\infty$ was done by Davie and Jewell [**2**] and the case $A = C(S)$ was done by Coburn [**1**].

(iii) Take μ to be two-dimensional Lebesgue measure on $[|z| < 1]$ and let H be the Bergman space of analytic $L^2(\mu)$ functions. We consider two possibilities for the algebra A.

$A = L^\infty$: We consider $\mathcal{M}$ to be the space of ultrafilters on the lattice of measurable sets modulo sets of measure zero. Then $\mathcal{T}/\mathcal{C} \approx C(\mathcal{M}_1)$ where $\mathcal{M}_1$ consists of those ultrafilters $\mathcal{E}$ having the property that if $E \in \mathcal{E}$ and $\varepsilon > 0$ there exists a "Carleson square" $S_h = \{z = re^{i\theta} | 1 - h < r < 1, |\theta - \theta_0| < h\}$ such that $\mu(E \cap S_h) > (1-\varepsilon)S_h$. The necessary calculations follow from a theorem of Hastings [**4**]; see also [**6**].

A equals the C^*-algebra generated by H^∞: In this case $\mathcal{M}$ is the maximal ideal space of H^∞ and $\mathcal{M}_1$ is the set of "one-point parts" of $\mathcal{M}$ (see [**5**]). This fact was established in [**6**]; the use of the present Theorem allows for a considerable simplification of the proof.

References

1. L. A. Coburn, *Singular integral operators and Toeplitz operators on odd spheres*, Indiana Univ. Math. J. **23** (1973), 433–439.
2. A. M. Davie and N. P. Jewell, *Toeplitz operators in several complex variables*, J. Funct. Anal. **26** (1977), 356–368.
3. R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
4. W. W. Hastings, *A Carleson measure theorem for Bergman spaces*, Proc. Amer. Math. Soc. **52** (1975), 237–241.
5. K. Hoffman, *Bounded analytic functions and Gleason parts*, Ann. of Math. (2) **86** (1967), 74–111.
6. G. McDonald and C. Sundberg, *Toeplitz operators on the disc*, Indiana Univ. Math. J. **28** (1979), 595–611.

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