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ON THE STRUCTURE OF SETS OF UNIQUENESS

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ABSTRACT. We show that every U_0-set is almost a W-set.

It may be expected that if a Borel set $E \subset \mathbf{T} \overset{\text{def}}{=} \mathbf{R}/\mathbf{Z}$ cannot carry any Borel measure μ whose Fourier-Stieltjes coefficients $\hat{\mu}(n) \overset{\text{def}}{=} \int_{\mathbf{T}} e^{-2\pi int}\, d\mu(t)$ vanish at infinity, then the arithmetic of E is partially responsible. We shall show that this is precisely the case.

Recall the following definitions (see [**3**]).

DEFINITION. A Borel measure μ on $\mathbf{T}$ is a *Rajchman measure* if $\lim_{|n|\to\infty} \hat{\mu}(n) = 0$; R denotes the set of Rajchman measures. A set E is a *set of uniqueness in the wide sense*, or U_0-*set*, if $\mu E = 0$ for all $\mu \in R$. A Borel set $E \subset \mathbf{T}$ is a W-*set* if there is some strictly increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that for all $x \in E$, $\{n_k x\}$ has a nonuniform asymptotic distribution ν_x.

Let us say that a set E is *almost in a class* $\mathcal{C}$ if for every positive Borel measure μ carried by E, there is a set $F \in \mathcal{C}$ such that $\mu(E \backslash F) = 0$. In [**3**], we showed that $\mu \in R$ if and only if $\mu E = 0$ for all $E \in W$. This immediately implies that every U_0-set is almost in W_σ, where W_σ is the class of sets which are countable unions of W-sets. Indeed, given $E \in U_0$ and μ a positive measure carried by E, we have that $\sup_{G\in W_\sigma} \mu G$ is attained. Since $\mu|_F \notin R$ for all Borel $F \subset E$ unless $\mu F = 0$, it is easy to see that $\sup_{G\in W_\sigma} \mu G = \|\mu\|$, whence the claim follows. We shall prove here the following stronger result.

THEOREM. *A Borel set E is a U_0-set if and only if E is almost a W-set.*

Of course, one direction is trivial since every W-set is a U_0-set. In the other direction, we shall prove a still stronger result. Recall [**3**] that E is a W_1-*set* if E is a W-set corresponding to asymptotic distributions ν_x with $\hat{\nu}_x(1) \neq 0$ for $x \in E$. We shall show that U_0-sets are in fact almost W_1-sets. Furthermore, with the definitions extended as in [**3**], U_0-sets are almost W_1-sets in all LCA groups. For related results, see [**1** and **2**].

LEMMA. *Let μ be a positive σ-finite measure. Suppose that f and g are measurable functions such that for every x, either $f(x) \neq 0$ or $g(x) \neq 0$. Then there exists a countable set $K \subset]0,\infty[$ such that if $\alpha \in]0,\infty[\backslash K$, then $f(x) + \alpha g(x) \neq 0$ for μ-a.e. x.*

PROOF. Let $G_\alpha = \{x\colon f(x) + \alpha g(x) = 0\}$. Then $G_\alpha \cap G_\beta = \varnothing$ if $\alpha \neq \beta$, whence $K = \{\alpha > 0\colon \mu G_\alpha > 0\}$ is at most countable. $\square$

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LEMMA. *Let μ be a positive σ-finite measure. Suppose that f_n are measurable functions bounded by* 1 *such that for every x, some $f_n(x)$ is not* 0. *Then there exist $\alpha_n > 0$ such that $\sum \alpha_n < \infty$ and $\sum \alpha_n f_n(x) \neq 0$ for μ-a.e. x.*

PROOF. It is easy to see that we may assume μ to be finite. Let $E_n = \{x\colon f_n(x) \neq 0\}$. We define α_n inductively. Let $\alpha_1 = 1$. If $\alpha_1, \ldots, \alpha_N$ have been defined, then choose $\alpha_{N+1} < \alpha_N/2$ such that $\sum_{n \leq N+1} \alpha_n f_n(x) \neq 0$ μ-a.e. on $\bigcup_{n \leq N+1} E_n$ and also

$$\mu\left(\left\{x \in \bigcup_{n \leq N} E_n\colon \left|\sum_{n \leq N} \alpha_n f_n(x)\right| < 2\alpha_{N+1}\right\}\right) < N^{-1}.$$

Then if $\sum_{n \geq 1} \alpha_n f_n(x) = 0$, we have for all N,

$$\left|\sum_{n \leq N} \alpha_n f_n(x)\right| = \left|\sum_{n > N} \alpha_n f_n(x)\right| \leq \sum_{n > N} |\alpha_n| < 2\alpha_{N+1},$$

whence

$$\mu\left(\left\{x\colon \sum_{n \geq 1} \alpha_n f_n(x) = 0\right\}\right) < N^{-1} + \mu\left(\left(\bigcup_{n \leq N} E_n\right)^c\right).$$

Since N is arbitrary, it follows that $\sum_{n \geq 1} \alpha_n f_n(x) \neq 0$ μ-a.e. □

REMARK. It is not hard to show by using Fubini's theorem that, in fact, almost all choices of $\{\alpha_n\}$ satisfy the lemma, where, say, α_n is chosen independently and uniformly in $]0, n^{-2}]$. One may also show that except for a meager set of positive sequences in $l^1(\mathbf{Z}^+)$, any positive sequence $\{\alpha_n\}$ satisfies the lemma.

PROOF OF THE THEOREM. Let E be a U_0-set and μ a positive Borel measure on E. Then by [**3**], there are W_1-sets E_m such that $\mu\left(E \setminus \bigcup_{m \geq 1} E_m\right) = 0$; such that if the sequence corresponding to E_m is $\{n_{k,m}\}$, then $\{n_{k,m}x\}$ has an asymptotic distribution $\nu_{m,x}$ μ-a.e.; and such that for all subsequences $\{n'_{k,m}\} \subset \{n_{k,m}\}$, $\{n'_{k,m}x\}$ also has the asymptotic distribution $\nu_{m,x}$ μ-a.e. Note that $\hat{\nu}_{m,x}(1) \neq 0$ for $x \in E_m$. By the lemma, we may choose $\{\alpha_m\}$ such that $\alpha_m > 0$, $\sum_{m \geq 1} \alpha_m = 1$, and $\sum_{m \geq 1} \alpha_m \hat{\nu}_{m,x}(1) \neq 0$ μ-a.e. Let $\{n_{k_i, m_i}\}_{i=1}^{\infty}$ be any strictly increasing sequence such that for all m,

$$\lim_{I \to \infty} \frac{1}{I} \operatorname{card}\{i \leq I\colon m_i = m\} = \alpha_m.$$

Then it is easy to see by Weyl's criterion that $\{n_{k_i, m_i} x\}$ has the asymptotic distribution $\sum \alpha_m \nu_{m,x}$ μ-a.e. with $(\sum \alpha_m \nu_{m,x})\hat{}(1) \neq 0$ μ-a.e. That is, $F = \{x\colon \{n_{k_i, m_i} x\}$ has an asymptotic distribution ν_x with $\hat{\nu}_x(1) \neq 0\}$ is a W_1-set such that $\mu(E \setminus F) = 0$. □

The extension to LCA groups is immediate, save for one subtlety. Namely, given a collection of sequences $\{\gamma_{k,m}\}_{k \geq 1} \subset \hat{G}$ $(m \geq 1)$ with $\lim_{k \to \infty} \gamma_{k,m} = \infty$, we must be able to mix subsequences of them (in appropriate proportions) so as to obtain a sequence still tending to ∞. This is achieved by an easy adaptation of the proof of Theorem 14 in [**3**].

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