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ON THE ZEROS OF POLYNOMIALS OF MINIMAL L_p-NORM

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ABSTRACT. It is shown in this note that the zeros of the minimal polynomials in the L_p-norm interlace with those of the Chebyshev polynomials of the first and second kinds.

Let us denote by $T_{n,p}(x)$ the algebraic polynomial of degree n with leading coefficient 1 which has the minimal L_p-norm on $[-1,1]$, $1 \leq p \leq \infty$. One of the most classical problems in approximation theory consists in determining these polynomials. The solution to this problem is well known in the cases when $p = 1, 2$, and ∞: $T_{n,1}(x) = U_n(x)$, the Chebyshev polynomial of second kind, $T_{n,2}(x) = L_n(x)$, the Legendre polynomial and $T_{n,\infty}(x) = \tilde{T}_n(x)$, the Chebyshev polynomial of first kind. The intriguing cases $1 < p < 2$ and $2 < p < \infty$ are still open despite numerous efforts made in this direction. The wide attention paid to Chebyshev polynomials is related to the fact that their zeros have many nice properties playing important roles in various applications. This raises the question of what can be said about the location of the zeros of $T_{n,p}(x)$ when $p \neq 1, 2, \infty$. Let us denote by $x_{i,n,p}$, $1 \leq i \leq n$, the zeros of $T_{n,p}$, where $-1 < x_{n,n,p} < \cdots < x_{1,n,p} < 1$. By a classical result of Markov and Stieltjes (see [**6**, p. 122]) the zeros of the Legendre polynomials interlace with those of the Chebyshev polynomials of second and first kinds; that is, $x_{i,n,1} < x_{i,n,2} < x_{i,n,\infty}$, $1 \leq i \leq [n/2]$. (Note that $T_{n,p}$ is even (odd) if n is even (odd), i.e., the location of zeros is symmetric about 0.) The main goal of this note is to extend the above statement to any p, $1 < p < \infty$; namely, to show that the zeros of $T_{n,p}$ interlace with those of U_n and $\tilde{T}_n$ for every $1 < p < \infty$.

Let us consider a family of positive weights $w(\tau, x)$ with parameter τ, $0 \leq \tau \leq 1$, such that $w(\tau, x) \in L_1[a,b]$ is absolutely continuous with respect to τ and x for $0 \leq \tau \leq 1$ and $a < x < b$, and

$$\sup_h \left| \frac{w(\tau + h, x) - w(\tau, x)}{h} \right| \in L_1[a,b] \cap C(a,b) \quad \text{for every } 0 \leq \tau \leq 1.$$

Let $\mathcal{T}_n(\tau, x)$ be the polynomial of degree n with leading coefficient 1 which has minimal L_p-norm on $[a,b]$ with respect to the weight $w(\tau, x)$, $1 \leq p \leq \infty$.

This polynomial is uniquely defined for every $1 \leq p \leq \infty$ and has n simple zeros $a < x_{n,p}(\tau) < \cdots < x_{1,p}(\tau) < b$. It is well known that for $1 \leq p < \infty$ $\mathcal{T}_n(\tau, x)$

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satisfies the following orthogonality relations:

$$(1) \qquad \int_a^b p_{n-1}(x)|\mathcal{T}_n(\tau,x)|^{p-1}\operatorname{sgn}\mathcal{T}_n(\tau,x)w(\tau,x)\,dx = 0, \qquad p_{n-1}\in P_{n-1},$$

where P_{n-1} denotes the set of all algebraic polynomials of degree at most $n-1$. (We shall disregard indices τ and p in notations for $x_{i,p}(\tau)$ and $\mathcal{T}_n(\tau,x)$, writing x_i and $\mathcal{T}_n$ instead when there is no possible confusion.)

Since $\mathcal{T}_n(x)=\prod_{i=1}^n(x-x_i)$ we can rewrite (1) as

$$(2) \qquad \int_a^b (x-x_i)^{-1}|\mathcal{T}_n(x)|^p w(\tau,x)\,dx = 0, \qquad 1\le i\le n.$$

Our first theorem shows that under a certain condition the x_i's are monotone functions of τ. For $p=2$, this statement was verified by Markov [**4**] by a different method.

THEOREM 1. *Let $1\le p<\infty$ and assume that*

$$\frac{\partial w(\tau,x)}{\partial\tau}\Big/ w(\tau,x)$$

is an increasing (decreasing) nonconstant function of $x\in(a,b)$ for every $0\le\tau\le 1$. Then $x_i(\tau)$ is a strictly increasing (decreasing) function of τ, $1\le i\le n$.

PROOF OF THEOREM 1. First of all let us note that if $\frac{\partial w}{\partial\tau}/w$ is an increasing nonconstant function then in view of (2), for every $1\le i\le n$, we have

$$(3) \qquad \int_a^b (x-x_i)^{-1}|\mathcal{T}_n(x)|^p\frac{\partial w(\tau,x)}{\partial\tau}\,dx$$
$$= \frac{1}{w(\tau,x_i)}\int_a^b |\mathcal{T}_n(x)|^p\left\{\frac{w(\tau,x_i)\frac{\partial w(\tau,x)}{\partial\tau}-\frac{\partial w(\tau,x_i)}{\partial\tau}w(\tau,x)}{x-x_i}\right\}dx>0.$$

Our proof will be based on the implicit function theorem, applied to the system (2) by setting

$$(4) \qquad F_k(x_1,\dots,x_n,\tau)=\int_a^b (x-x_k)^{-1}|\mathcal{T}_n(x)|^p w(\tau,x)\,dx = 0, \qquad 1\le k\le n.$$

Then the $x_i(\tau)$'s are C^1-functions of τ, provided that

$$\Delta_0=\frac{\partial(F_1,\dots,F_n)}{\partial(x_1,\dots,x_n)}\ne 0.$$

In this case $dx_i/d\tau=-\Delta_i/\Delta_0$, where

$$\Delta_i=\frac{\partial(F_1,\dots,F_i,\dots,F_n)}{\partial(x_1,\dots,\tau,\dots,x_n)}, \qquad 1\le i\le n.$$

In order to find these determinants we calculate the corresponding partial derivatives. Since $(x-x_k)^{-1}\partial\mathcal{T}_n/\partial x_j\in P_{n-2}$ if $j\ne k$ it follows from (4) and (1) that for every p, $1\le p<\infty$, and $j\ne k$

$$(5) \qquad \frac{\partial F_k}{\partial x_j}=\int_a^b p|\mathcal{T}_n(x)|^{p-1}\operatorname{sgn}\mathcal{T}_n(x)\frac{(\partial\mathcal{T}_n(x)/\partial x_j)}{x-x_k}w(\tau,x)\,dx=0.$$

Evidently,

$$\frac{\partial F_k}{\partial \tau} = \int_a^b (x - x_k)^{-1} |\mathcal{T}_n(x)|^p \frac{\partial w(\tau, x)}{\partial \tau}\, dx. \tag{6}$$

Furthermore, if $1 < p < \infty$,

$$\begin{aligned}\frac{\partial F_k}{\partial x_k} &= \int_a^b \left|\frac{\mathcal{T}_n(x)}{x - x_k}\right|^p \frac{\partial}{\partial x_k} \left(|x - x_k|^{p-1} \operatorname{sgn}(x - x_k)\right) w(\tau, x)\, dx \\ &= (1-p) \int_a^b (x - x_k)^{-2} |\mathcal{T}_n(x)|^p w(\tau, x)\, dx,\end{aligned} \tag{7}$$

while for $p = 1$ we get

$$\begin{aligned}\frac{\partial F_k}{\partial x_k} &= \frac{\partial}{\partial x_k} \left\{ -\int_a^{x_k} \left|\frac{\mathcal{T}_n(x)}{x - x_k}\right| w(\tau, x)\, dx + \int_{x_k}^b \left|\frac{\mathcal{T}_n(x)}{x - x_k}\right| w(\tau, x)\, dx \right\} \\ &= -2|\mathcal{T}_n'(x_k)| w(\tau, x_k).\end{aligned}$$

(This last differentiation and (5) are straightforward; (6) follows from conditions imposed on $w(\tau, x)$ and Lebesgue's theorem on dominated convergence; (7) is a special case of differentiating convolutions, see [**2**, p. 55].)

Since Δ_0 is a diagonal determinant we get

$$\Delta_0 = \prod_{k=1}^n \frac{\partial F_k}{\partial x_k}, \qquad \Delta_i = \frac{\partial F_i}{\partial \tau} \prod_{\substack{k=1 \\ k \neq i}}^n \frac{\partial F_k}{\partial x_k}.$$

Thus, for $1 < p < \infty$,

$$\frac{dx_i}{d\tau} = -\frac{\partial F_i / \partial \tau}{\partial F_i / \partial x_i} = \frac{\displaystyle\int_a^b (x - x_i)^{-1} |\mathcal{T}_n(x)|^p \frac{\partial w(\tau, x)}{\partial \tau}\, dx}{\displaystyle(p-1) \int_a^b (x - x_i)^{-2} |\mathcal{T}_n(x)|^p w(\tau, x)\, dx}$$

while for $p = 1$

$$\frac{dx_i}{d\tau} = -\frac{\partial F_i / \partial \tau}{\partial F_i / \partial x_i} = \frac{\displaystyle\int_a^b (x - x_i)^{-1} |\mathcal{T}_n(x)| \frac{\partial w(\tau, x)}{\partial \tau}\, dx}{2|\mathcal{T}_n'(x_k)| w(\tau, x_i)}.$$

Finally, taking into account inequalities in (3), we complete the proof of the theorem.

COROLLARY 1. *Let $p = \infty$ and let $w(\tau, x) \in C[a, b]$ be as in Theorem* 1. *Then $x_{i,\infty}(\tau)$ is an increasing (decreasing) function of τ, $1 \le i \le n$.*

PROOF. We can apply Theorem 1 for the family of weights $\tilde{w}(\tau, x) = w^p(\tau, x)$, since $\frac{\partial \tilde{w}}{\partial \tau} / \tilde{w} = p \frac{\partial w}{\partial \tau} / w$. On the other hand the L_p-norm with weight $w^p(\tau, x)$ tends to the L_∞-norm with weight $w(\tau, x)$, implying that (see [**3**] for details) for each $0 \le \tau \le 1$, $\tilde{x}_{i,p} \to x_{i,\infty}$ as $p \to \infty$ $(1 \le i \le n)$ where $\tilde{x}_{i,p}$ corresponds to the weight $\tilde{w}(\tau, x) = w^p(\tau, x)$. This verifies the corollary.

Assume now that the weights $w(\tau, x)$ are even functions of $x \in [-1, 1]$ for every $0 \le \tau \le 1$. Then, obviously $\mathcal{T}_n$ is even (odd) if n is even (odd). Therefore in this case the zeros of $\mathcal{T}_n$ are symmetric about 0, i.e. it suffices to consider them on $[0, 1]$.

COROLLARY 2. *Set* $[a,b]=[-1,1]$, $1\le p<\infty$ *and assume that each* $w(\tau,x)$ *is an even function of* x *on* $[-1,1]$, *such that*

$$\frac{\partial w(\tau,x)}{\partial \tau}\Big/ w(\tau,x)$$

is an increasing (decreasing) function of $x\in(0,1)$ *for every* $0\le\tau\le 1$. *Then for every* $1\le i\le[n/2]$, $x_i(\tau)$ *is a strictly increasing (decreasing) function of* τ.

PROOF. Assume that $n=2m$ (the case of odd n is similar). Transforming (1) to $[0,1]$, we obtain

$$\int_0^1 x^{2k}|\mathcal{T}_n(x)|^{p-1}\operatorname{sgn}\mathcal{T}_n(x)w(\tau,x)\,dx=0,\qquad 0\le k\le m-1,$$

where $\mathcal{T}_n(x)=\prod_{i=1}^m(x^2-x_i^2)$. With a proper substitution this yields

$$\int_0^1 x^{k}|\mathcal{T}_m^*(x)|^{p-1}\operatorname{sgn}\mathcal{T}_m^*(x)w^*(\tau,x)\,dx=0,\qquad 0\le k\le m-1, \tag{8}$$

where $\mathcal{T}_m^*(x)=\prod_{i=1}^m(x-x_i^*)$, $x_i^*=x_i^2$, $1\le i\le m$, and $w^*(\tau,x)=w(\tau,\sqrt{x})/\sqrt{x}$. Evidently,

$$\frac{\partial w^*(\tau,x)}{\partial \tau}\Big/ w^*(\tau,x)$$

is increasing (decreasing) for $x\in(0,1)$, and by (8) $\mathcal{T}_m^*$ is the minimal polynomial of degree m on $[0,1]$ with respect to the L_p-norm with weight $w^*(\tau,x)$. Thus, applying Theorem 1 for $[a,b]=[0,1]$, $n=m$, and $w(\tau,x)=w^*(\tau,x)$, we obtain Corollary 2.

Now we can state our main result.

THEOREM 2. *Let* $-1<x_{n,n,p}<x_{n-1,n,p}<\cdots<x_{1,n,p}<1$ *be the zeros of the polynomial* $T_{n,p}(x)=x^n+\cdots$ *having minimal* L_p*-norm on* $[-1,1]$ $(1\le p\le\infty)$. *Then* $x_{i,n,1}<x_{i,n,p}<x_{i,n,\infty}$ *for every* $1\le i\le[n/2]$ *and* $1<p<\infty$.

PROOF. Let w_1 and w_2 be positive even weights absolutely continuous on $(-1,1)$ and assume that w_1/w_2 is nonconstant and increasing (decreasing) on $(0,1)$. Set $w(\tau,x)=\tau w_1(x)+(1-\tau)w_2(x)$, $0\le\tau\le1$. Then

$$\frac{\partial w(\tau,x)}{\partial \tau}\Big/ w(\tau,x)$$

is an increasing (decreasing) function of $x\in(0,1)$ for every $0\le\tau\le1$. Hence, by Corollary 2, $x_i(0)<x_i(1)$ (respectively, $x_i(0)>x_i(1)$), $1\le i\le[n/2]$, where $x_i(0)$ and $x_i(1)$ are the zeros of polynomials $\mathcal{T}_n$ and $\mathcal{T}_n^*$ satisfying (1) with weights $w(0,x)=w_2(x)$ and $w(1,x)=w_1(x)$, respectively. Let us consider the case where $w_1(x)=(1-x^2)^{-1/2}$ and $w_2(x)\equiv1$. Then obviously $\mathcal{T}_n=T_{n,p}$, the Chebyshev polynomial in the L_p-norm, and w_1/w_2 is increasing on $(0,1)$. It is known (see [**1**, p. 251]) that $T_{n,\infty}=\tilde{T}_n$ satisfies the orthogonality relations (1) with weight $(1-x^2)^{-1/2}$ for every $1<p<\infty$, i.e. $\mathcal{T}_n^*=T_{n,\infty}$ in this case. Thus we obtain $x_{i,n,p}<x_{i,n,\infty}$ for every $1\le i\le[n/2]$ and $1<p<\infty$. Finally, by setting $w_1(x)=(1-x^2)^{(p-1)/2}$ and $w_2(x)\equiv1$, we ensure that w_1/w_2 is decreasing on $(0,1)$ and therefore $x_i(0)>x_i(1)$ $(1\le i\le[n/2])$. It is shown in [**1**, p. 251] that $T_{n,1}=U_n$ satisfies (1) with weight $(1-x^2)^{(p-1)/2}=w_1(x)$. Thus, $\mathcal{T}_n=T_{n,p}$ and

$\mathcal{T}_n^* = T_{n,1}$ in this case, implying that $x_{i,n,p} > x_{i,n,1}$ $(1 \leq i \leq [n/2],\ 1 < p < \infty)$. This completes the proof of Theorem 2.

As an immediate consequence of this theorem, we note that the absolute values of the coefficients of the minimal polynomials in L_p-norm are larger (respectively, smaller) than the absolute values of those of the Chebyshev polynomials of the second (respectively, first) kind.

Let us mention another application of Theorem 2. For a function $f \in C[-1,1]$ denote by $e_{n-1}(f)_p = f - q_{n-1}(f)_p$ its error function in L_p-approximation by algebraic polynomials of degree at most $n-1$ ($q_{n-1}(f)_p$ is the best L_p-approximant). It is shown in [**5**] that if f is a generalized convex function of degree n (that is, it forms a Chebyshev system with polynomials of degree n), then the zeros of $e_{n-1}(f)_p$ and $T_{n,p}$ interlace. Now using Theorem 2 we also have a fairly precise description of distribution of the zeros of $e_{n-1}(f)_p$ for functions f with the above property.

Finally, we would like to mention an open problem. Using our approach, it can be easily shown that the $x_{i,n,p}$'s (zeros of $T_{n,p}$) are C^1-functions of p as p varies from 1 to $+\infty$. We believe that for every $1 \leq i \leq [n/2]$ $x_{i,n,p}$ is a *strictly increasing* function of $p \in [1, \infty)$.

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