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CLOSED SUBSPACES OF FINITE CODIMENSION IN SOME FUNCTION ALGEBRAS

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ABSTRACT. We characterize all closed subspaces of finite codimension in some specific types of function algebras e.g. these include $C(X)$: algebra of continous functions on a compact Hausdorff space, $C^n[a,b]$: the algebra of n-times continuously differentiable functions on the closed interval $[a,b]$. Our work is a generalization of the well-known Gleason-Kahane-Želazko theorem [**3**, **6**] for subspaces of codimension one in arbitrary unitary Banach algebras.

0. Introduction. In [**5**], K. Jarosz settled a conjecture of Warner and Whitley [**10**] by proving the following: Let S be a compact Hausdorff space such that each point of S is a G_δ-set and M be a closed subspace of finite codimension in $\mathcal{C}(S)$, the algebra of continuous functions on S with the sup norm. Let k be any positive integer. Suppose that for every f in M, f has a least k distinct zeroes in S. Then there exist k distinct points $s_1, s_2, \ldots, s_k$ in S such that for every f in M; $f(s_1) = f(s_2) = \cdots = f(s_k) = 0$.

In this paper, we generalize this theorem as follows.

THEOREM 3.1. *Let X be a compact Hausdorff space and M be a closed subspace of finite codimension in $\mathcal{C}(X)$ such that for every f in M, $f(x) = 0$ for some x in X. Then* (a) *There exists an x_0 in X such that $f(x_0) = 0$ for every f in M;* (b) *there exists finitely many points $x_1, x_2, \ldots, x_k$ such that $f(x_j) = 0$ for $1 \leq j \leq k$ and these are the only common zeroes of M in the following sense: If C is any compact G_δ-set containing $x_1, x_2, \ldots, x_k$, then there exists an f in M such that $f(x) \neq 0$ for every x in $X \setminus C$.*

Also, in this paper, we prove the polynomial lemma.

LEMMA 1.1. *Let K be any closed subset of the complex plane such that $K^0 = \varnothing$. Let p_j, $1 \leq j \leq k$, be nonconstant polynomials of one complex variable z. Further assume $\sum_{j=1}^k w_j p_j$ has a zero in K for any given complex numbers w_j, $1 \leq j \leq k$. Then there exists a z_0 in K such that $p_j(z_0) = 0$ for $1 \leq j \leq k$.*

This is a generalization of a lemma in [**5**] and this allows us to obtain theorems of type 3.1 for algebras $\mathcal{C}^n[a,b], \mathcal{L}^1(\mathcal{R})$ as well as $\mathcal{C}(X)$. Recently C. P. Chen and P. J. Cohen [**2**] have proved that conclusions of Theorem 3.1 hold for selfadjoint regular n-point spectral Banach algebras. These include $\mathcal{C}(X)$ [**5**] and also $L^1(G)$ where G is any locally compact metrizable abelian group. Thus this solves both

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conjectures of Warner and Whitley [**10**]. But [**2** and **5**] do not cover algebras of type $\mathcal{C}^n[a,b]$, because these are not spectral. One of us in [**8**] has removed the condition of spectrality in the Chen and Cohen theorem, which then applies to algebras of n-times continuously differentiable functions. The results in this paper have some overlap with results from [**2**, **5** and **8**], but the method given here is different and more direct. Further, our results apply to a more general class of algebras e.g., Fréchet algebras. For related results, refer to [**1**, **3**, **6** and **9**].

1. Proof of the polynomial Lemma 1.1 (as stated in the introduction). Suppose that $p_j(z)$, $1 \le j \le k$, have no common zeroes in K. If $p_0(z)$ is the greatest common divisor of p_j, $1 \le j \le k$; by our supposition, p_0 does not vanish on K. Thus replacing p_j by p_j/p_0, we can assume that any complex linear combination of p_j has a zero in K but $\{p_j\}$ do not have a common zero in the complex plane.

Let $\mathcal{P}(z,w) = \sum_{j=1}^k w_j p_j(z)$ where w denotes the k tuple $(w_1, w_2, \ldots, w_k)$; R denotes the polynomial ring over $\mathbf{C}$ in k indeterminates $w_1, w_2, \ldots, w_k$; Q be the quotient field of R; and $R[z]$, $Q[z]$ be the polynomial rings in one indeterminate z over the rings R and Q respectively.

We claim that $\mathcal{P}(z,w)$ is irreducible in the ring $R[z]$. If not, there exist polynomials $\mathcal{P}_1(z,w), \mathcal{P}_2(z,w)$ nonconstant and belonging to $R[z]$ so that $\mathcal{P}(z,w) = \mathcal{P}_1(z,w)\mathcal{P}_2(z,w)$. We can write

$$\mathcal{P}_i(z,w) = \sum_{0 \le |j| \le N} p_{ij} w^j$$

where $w^j = w_1^{j_1} w_2^{j_2} \cdots w_k^{j_k}$, j is the multi-index $(j_1, j_2, \ldots, j_k)$, $|j| = \sum_{\lambda=1}^k j_\lambda$, $p_{i,j}(z)$ is a polynomial in z with constant coefficients for $i = 1, 2$. Thus we consider $\mathcal{P}_i(z,w)$ as polynomials in w over the integral domain $\mathbf{C}[z]$. Since $\mathcal{P}(z,w)$ is a linear polynomial in w_j; we see that at least one of $\mathcal{P}_1, \mathcal{P}_2$ must be independent of w. Let us assume $\mathcal{P}_1$ to be independent of w and P_2 to be linear in w.

Let $\mathcal{P}_1(z) = \mathcal{P}_1(z,w)$. Thus $p_j(z) = \mathcal{P}_1(z) \cdot p_{2,j}(z)$, $1 \le j \le k$, that is, $p_j(z)$ have a common divisor $\mathcal{P}_1(z)$. By our assumption, they do not have any common divisor except constants and on the other hand $\mathcal{P}_1(z) = \mathcal{P}_1(z,w)$ is nonconstant, which is a contradicition. Hence $P(z,w)$ is irreducible. Also, $\mathcal{P}(z,w)$ is primitive as a polynomial over R by the same reasoning.

We now show that $\mathcal{P}(z,w)$ is irreducible in the ring $Q[z]$. If not, then $\mathcal{P}(z,w) = \mathcal{P}_1(z,w)\mathcal{P}_2(z,w)$ where $\mathcal{P}_1, \mathcal{P}_2$ are nonconstant polynomials in z with coefficients in Q. Since R is a unique factorization domain [**11**, p. 45] we can find $\alpha_1, \alpha_2, \beta_1, \beta_2$ in R such that α_1, α_2 are coprime; β_1, β_2 are coprime and

$$\frac{\alpha_1}{\alpha_2}\mathcal{P}_1, \quad \frac{\beta_1}{\beta_2}\mathcal{P} \text{ belong to } R[z]$$

and are primitive polynomials. By Gauss' Lemma [**11**, p. 46], the product of primitive polynomials is primitive and so

$$\frac{\alpha_1\beta_1}{\alpha_2\beta_2}\mathcal{P}(z,w)$$

is a primitive polynomial. Since $\mathcal{P}$ is also primitive, we see that $\alpha_1\beta_1/\alpha_2\beta_2$ is a unit and since $\alpha_1\beta_1, \alpha_2\beta_2$ are coprime, all of $\alpha_1, \beta_1, \alpha_2, \beta_2$ are units in the ring R

i.e. they are constants. This means that $\mathcal{P}_1, \mathcal{P}_2$ belong to $R[z]$. But $\mathcal{P} = \mathcal{P}_1\mathcal{P}_2$ is irreducible in $R[z]$, which is a contradiction. Hence $\mathcal{P}$ is irreducible in $Q[z]$. But then since $Q[z]$ is a Euclidean domain there exist A, B in $Q[z]$ such that

$$A\mathcal{P} + B\frac{\partial \mathcal{P}}{\partial z} = 1.$$

Thus there exists a nonzero D in R such that DA, DB belong to $R[z]$ and

$$DAP + DB\frac{\partial \mathcal{P}}{\partial z} = D$$

i.e.

$$A_1P + B_1\frac{\partial \mathcal{P}}{\partial z} = D.$$

Since D is a polynomial in w_j, we can choose a ball Ω in $\mathbf{C}^k$ on which $D(w) \neq 0$. Thus for every w in Ω, the polynomial $\mathcal{P}(z, w)$ has k distinct roots $z_j(w)$, $1 \leq j \leq k$, and we may assume that z_j are holomorphic in Ω [**4**, p. 11]. Let $S_j = \{\omega \in \Omega; z_j(\omega) \in K\}$. One of the z_j belongs to K for any given w in Ω. So $\Omega = \bigcup S_j$. Since each S_j is closed in Ω, one of the S_j say S_1, must have nonempty interior. By the open mapping theorem, we see that $z_1(w)$ is locally constant on S_1 or $z_1(S_1)$ has nonempty interior. But $K^0 = \emptyset$. Thus z_1 is constant on Ω. Hence there exists a z_0 such that $P(z_0, w) \equiv 0$ on Ω and by analytic continuation on all of $\mathbf{C}^k$. Thus $p_j(z_0) = 0$ for $1 \leq j \leq k$, which is a contradiction. This proves Lemma 1.1.

COROLLARY 1.2. *If M is any subspace of the polynomial algebra $\mathbf{C}[z]$ and Z is the set of common zeroes of M allowing multiplicity, then there exists a polynomial p in M such that $p \neq 0$ on $K \setminus Z$.*

PROOF. Suppose that for every p in M, p has a zero in K. Let Z_p be the set of zeroes of p in K counting multiplicity. Then by Lemma 1.1, the family of $\{Z_p; p \in M\}$ is family of compact sets with the finite intersection property. Hence $Z = \bigcap_{p\in M} Z_p \neq \emptyset$. Let $Z = \{z_j, 1 \leq j \leq k\}$ and $q = (z - z_1)(z - z_2)\cdots(z - z_k)$; $M_0 = \{p \in \mathbf{C}[z]; qp \in M\}$. Now if every p in M_0 has a zero in K, once again by Lemma 1.1, there exists a z_0 in K such that $p(z_0) = 0$ for every p in M_0. But then z_0 is a common zero of M and so z_0 belongs to Z. This increases the multiplicity of z_0 for M, which is a contradiction. Thus there exists a p_0 in M_0 such that $p_0 \neq 0$ on K and qp_0 belongs to M and $qp_0 \neq 0$ in $K \setminus Z$.

2. Applications.

THEOREM 2.1. *Let K be a closed subset of C such that $K^0 = \emptyset$. Let $\mathcal{A}$ be an algebra of functions on K provided with a locally convex Hausdorff topological vector space structure in which evaluation at any point in K is continuous on $\mathcal{A}$. Suppose that M is a subspace of $\mathcal{A}$ and $\mathcal{P}$ is the algebra of polynomials in z. If for every f in M, there exists a z in K such that $f(z) = 0$ and $\mathcal{P} \cap M$ is dense in M; then the set Z of common zeroes of M is finite and nonempty, and there exists an f in M such that $f \neq 0$ on $K \setminus Z$.*

PROOF. By our assumptions and the polynomial lemma, we obtain that functions in $\mathcal{P} \cap M$ have at least one common zero. Since point evaluations are continuous and $\mathcal{P} \cap M$ is dense in M, all functions in M have at least one common zero.

So the set Z of common zeroes of M is finite and nonempty. Further, by Corollary 1.2, there exists a p in $\mathcal{P} \cap M$ such that $p \neq 0$ on $K \setminus Z$. Q.E.D.

COROLLARY 2.2. *Let $K = [a, b]$, a closed interval in the real line, and $\mathcal{C}^n(K)$ be the algebra of n-times continuously differentiable functions with the standard Banach algebra structure. Suppose M is a closed subspace of $\mathcal{C}^n(K)$ of finite codimension. Then the set Z of all common zeroes of M is finite (possibily empty) and there exists a fucntion f in M such that $f \neq 0$ in $K \setminus Z$.*

PROOF. Let $\mathcal{P}$ be the algebra of polynomials in z. Since M is of finite codimension and $\mathcal{P}$ is dense in $\mathcal{C}^n(K)$, $\mathcal{P} \cap M$ is dense in M. Applying Theorem 2.1, we obtain the result.

REMARK 2.3. The proof above also applies to $\mathcal{C}^\infty(R)$, the Fréchet algebra of $\mathcal{C}^\infty$ functions on $\mathcal{R}$.

COROLLARY 2.4. *Let $\mathcal{A}$ be the algebra of functions on $\mathcal{R}$ which are the Fourier transforms of L^1-functions on $\mathcal{R}$. If $f = \hat{g}$ is in $\mathcal{A}$, we define $\|f\| = \|g\|_1$; A is a Banach algebra. Let M be a closed subspace of finite codimension in $\mathcal{A}$. If Z is the set of common zeroes of M, then Z is finite (possibly empty) and there exists an f in M such that $f \neq 0$ in $R \setminus Z$.*

PROOF. Let $\mathcal{P} = \{p \text{ a polynomial}; p(t)e^{-t^2/2} \in M\}$ and $\mathcal{D} = e^{-t^2/2}P$. It is well known that $\mathcal{D}$ is a dense subspace of $\mathcal{A}$ and hence $\mathcal{D} \cap M$ is dense in M. Thus t is a common zero of M if and only if it is a common zero of $\mathcal{D} \cap M$ and hence of $\mathcal{P}$. But by the polynomial lemma, it follows that if Z is the set of common zeroes of $\mathcal{P}$ allowing multiplicity, then Z is finite (possibly empty) and there exists p in P such that $p(t) \neq 0$ on $R \setminus Z$. Q.E.D.

3. Proof of Theorem 3.1 (see the introduction for the formulation). By Theorem 1 of Jarosz [**5**], all functions in M have a common zero in X. Since M is of finite codimension and closed, there can only be finitely many common zeroes (say) $x_1, x_2, \ldots, x_k$. Also there exist finitely many Borel measures μ_j, $1 \leq j \leq l$, of bounded variation on X such that f belongs to M if and only if $f(x_j) = 0$ for $1 \leq j \leq k$ and $\int f\, d\mu_j = 0$ for $1 \leq j \leq l$.

We may assume that $|\mu_j|(Z) = 0$ for $1 \leq j \leq l$ where Z is the set of all zeroes of M. Let

$$\tilde{M} = \left\{ f \in \mathcal{C}(X); \int f\, d\mu_j = 0 \text{ for } 1 \leq j \leq l \right\}.$$

By Theorem 1 of [**5**], either there exists an f in $\tilde{M}$ such that $f \neq 0$ on X or all functions in $\tilde{M}$ have a common zero x_0. But such an x_0 would belong to Z and the Dirac measure δ_{x_0} would belong to the linear span of μ_j, $1 \leq j \leq l$. This is impossible since δ_x, x in Z, and μ_j, $1 \leq j \leq l$, are linearly independent. Hence there exists an f in M such that $f \neq 0$ on X.

Now let us choose f_j so that $f_j(Z) = 0$ and $\int f_i\, d\mu_j = \delta_{ij}$, $1 \leq i, j \leq l$. Choose an $\varepsilon > 0$ such that $|f| - \varepsilon \sum |f_j| > 0$ on X and an open neighborhood V of Z such that

$$|\mu_j|(V) < \varepsilon / \operatorname*{Max}_x \left(|f| - \varepsilon \sum |f_j| \right).$$

Choose a compact G_δ set C_0 such that $Z \subset C_0 \subset V$ and a function g in $\mathcal{C}(X)$ such that $0 \leq g \leq 1$, $g = 1$ only on $C_0, g = 0$ outside V. This is possible only because C_0 is a G_δ set.

Let $\psi = g(|f| - \varepsilon \sum |f_j|)$. Then

$$\begin{aligned} |\mu_j \psi| &\le |\mu_j|(V) \operatorname{Max} |\psi| \\ &< \left(\varepsilon / \operatorname{Max}\left(|f| - \varepsilon \sum |f_j|\right)\right) \cdot \operatorname{Max}\left(|f| - \varepsilon \sum |f_j|\right) \\ &\le \varepsilon. \end{aligned}$$

Consequently $f - \psi + \sum \mu_j(\psi) f_j = h$ satisfies

$$|h| \ge |f| - |\psi| - \varepsilon \sum |f_j| = (1 - g)\left(|f| - \varepsilon \sum |f_j|\right)$$

and so $|h| > 0$ on $X \setminus C_0$. Further $h(Z) = 0$, $\int h \, d\mu_j = -\mu_j(\psi) + \mu_j(\psi) = 0$ for $1 \le j \le l$. So h belongs to M. Q.E.D.

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