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APPROXIMATE INNERNESS OF POSITIVE LINEAR MAPS OF FINITE VON NEUMANN ALGEBRAS. II

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ABSTRACT. Let M be a σ-finite, finite von Neumann algebra with a faithful, normalized normal trace Tr on M. Let ρ be a positive linear map of M into itself such that $\rho(1)$ is not necessarily a projection. If ρ is approximately inner with respect to the norm $\|\cdot\|_2$ induced by Tr, then ρ has a close connection to *-homomorphisms.

1. Introduction. We showed the following theorem in our former paper [**5**]. Let M be a σ-finite, finite von Neumann algebra with a faithful, normalized normal trace Tr. Let ρ be a positive linear map of M into itself and approximately inner with respect to a net $\{a_\lambda\}$ such that $\|a_\lambda^* a_\lambda - f\|_2 \to 0$ and $\|a_\lambda a_\lambda^* - e\|_2 \to 0$ for two projections e and f in M where $\|x\|_2 = \mathrm{Tr}(x^*x)^{1/2}$ for every $x \in M$. Then ρ is a *-homomorphism of the von Neumann algebra eMe to the von Neumann algebra fMf.

The above-mentioned result was shown in the case of the unital positive linear maps in the sense that $\rho(1)$ is a projection. Thus, we shall examine a similar result in the case that a positive linear map ρ is not necessarily unital, that is, $\rho(1)$ is not necessarily a projection. Before we start the arguments, we set down some known results and some notation used in this paper.

Throughout this paper, we shall use the following notation: Let M be a σ-finite, finite von Neumann algebra and Tr a (fixed) faithful, normalized normal trace on M. Let $\|\cdot\|_2$ be the norm on M defined by $\|x\|_2 = \mathrm{Tr}(x^*x)^{1/2}$ for every $x \in M$.

Many authors (for example [**1**, **4**, **6**, **7**]) studied the completely positive maps of C^*-algebras and their results shall give useful roles for our paper. Thus, we recall the definition of the completely positive maps of C^*-algebras. Let A and B be C^*-algebras and n a natural number. A linear map ρ of A to B is said to be n-positive if the multiplicity map ρ_n from the matrix C^*-algebra $M_n(A)$ over A to the C^*-algebra $M_n(B)$ over B defined by $\rho_n([a_{ij}]) = [\rho(a_{ij})]$ is a positive map. If ρ is n-positive for every n, we call ρ completely positive.

In this paper we shall examine when a positive linear map of a von Neumann algebra M into itself is close to a *-homomorphism. Then such a positive linear map is a completely positive linear map. Thus, we recall the definition of positive linear maps which was introduced in [**5**].

DEFINITION 1. Let M be a σ-finite, finite von Neumann algebra with a faithful, normalized normal trace Tr on M. A positive linear map ρ of M into itself is

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approximately inner if there exists a net $\{a_\lambda\}$ in M satisfying $\lim \|\rho(x) - a_\lambda^* x a_\lambda\|_2 = 0$ for every $x \in M$.

Under the above definition, if ρ is a positive linear map and approximately inner, then ρ is a completely positive map of M to M. This was shown in the remark after Definition 2 in [**5**] by using [**6**, Chapter IV, Corollary 3.4]. We assumed in [**5**] that the net $\{a_\lambda\}$ is not necessarily bounded, but we shall assume throughout this paper that every net satisfying the conditions in the above definition is bounded.

We shall use two books [**4** and **6**] for the fundamental properties in the theory of von Neumann algebras and C^*-algebras.

2. Results. We shall give a generalization of Theorem 3 in [**5**]. Before we show the main theorem, we give some considerations. We deal with nonunital completely positive maps in this paper and so we use the following property which appears in [**7**]. The reference [**7**] was written in Japanese and so we give the proof here.

LEMMA 2. *Let A be a unital C^*-algebra and N a von Neumann algebra acting on a Hilbert space H. Let ρ be a completely positive linear map of A to N such that $\rho(1) = c$ and the support projection of c is f $(= \mathrm{supp}(c))$. Then there exists a completely positive linear map π of A to N such that $\pi(1) = f$ and $\rho(x) = c^{1/2}\pi(x)c^{1/2}$ for every $x \in A$.*

PROOF. For each natural number n, put

$$\pi_n(x) = (c + 1/n)^{-1/2}\rho(x)(c + 1/n)^{-1/2}$$

for every $x \in A$. Then π_n are completely positive maps of A to N. We shall show that, for every $x \in A$, the sequence $\{\pi_n(x)\}$ converges in the strong topology. Since the sequence $\{c(c + 1/n)^{-1}\}$ converges to f in the strong topology and $b_n = f - c(c + 1/n)^{-1}$ are positive elements, the sequence $\{b_n^{1/2}\}$ converges to 0 in the strong topology. Furthermore, the inequality

$$b_n^{1/2} \geq f - c^{1/2}(c + 1/n)^{-1/2} \geq 0$$

induces the relation; the strong-limit of $\{c^{1/2}(c+1/n)^{-1/2}\} = f$. Given an arbitrary element a of A with $0 \leq a \leq 1$, then $0 \leq \rho(a) \leq c$ and so $\rho(a)^{1/2} \leq c^{1/2}$. Thus, there exists an element y in $B(H)$ such that $\rho(a)^{1/2} = yc^{1/2} = c^{1/2}y^*$ where $B(H)$ is the von Neumann algebra of all bounded operators on H. Hence, the sequences $\{\rho(a)^{1/2}(c+1/n)^{-1/2}\}$ and $\{(c+1/n)^{-1/2}\rho(a)^{1/2}\}$ are bounded and strongly convergent sequences. The above properties imply that the sequence $\{\pi_n(a)\}$ is strongly convergent. Therefore, for every $x \in A$, $\{\pi_n(x)\}$ is strongly convergent and if we put $\pi(x) =$ the strong-limit of $\{\pi_n(x)\}$, the map π is the desired completely positive map. Q.E.D.

Let a be an element of M and $a = v|a|$ the polar decomposition of a. Since M is a finite von Neumann algebra, there exists a unitary element u in M satisfying $a = u|a|$. Thus, we can show that the net $\{a_\lambda\}$ appearing in Definition 1 can be replaced by another net of the following form.

LEMMA 3. *Let ρ be a positive linear map of M to itself such that $\rho(1) = c$. Let $\{a_\lambda\}$ be a bounded net in M with $a_\lambda = v_\lambda|a_\lambda| = u_\lambda|a_\lambda|$ where $v_\lambda|a_\lambda|$ is the polar decomposition of a_λ and u_λ is a unitary element having the property $u_\lambda f = v_\lambda$ for*

$\mathrm{supp}(c) = f$. *Suppose that* ρ *is approximately inner with respect to the net* $\{a_\lambda\}$. *Then* ρ *is approximately inner with respect to the net* $\{u_\lambda c^{1/2}\}$ *and also* $\{v_\lambda c^{1/2}\}$.

PROOF. By a result of Haagerup [**2**, Lemma 2.10], we can show the inequality

$$\| |x| - |y| \|_2 \leq \| |x|^2 - |y|^2 \|_1$$

for any $x, y \in M$ where $\|a\|_1 = \mathrm{Tr}(|a|)$ for $a \in M$. Thus, we have the following relation:

$$\begin{aligned} \| |a_\lambda| - c^{1/2}\|_2^2 &\leq \| |a_\lambda|^2 - c\|_1 = \|a_\lambda^* a_\lambda - c\|_1 \\ &= \mathrm{Tr}(|a_\lambda^* a_\lambda - c|) \leq \mathrm{Tr}((a_\lambda^* a_\lambda - c)^2)^{1/2} = \|a_\lambda^* a_\lambda - c\|_2. \end{aligned}$$

Since $\lim \|c - a_\lambda^* a_\lambda\|_2 = \lim \|\rho(1) - a_\lambda^* a_\lambda\|_2 = 0$, we have the relation

$$\lim \| |a_\lambda| - c^{1/2}\|_2 = 0.$$

Furthermore, we have the relation

$$\begin{aligned} \lim \|a_\lambda - u_\lambda c^{1/2}\|_2 &= \lim \|u_\lambda |a_\lambda| - u_\lambda c^{1/2}\|_2 \\ &= \lim \| |a_\lambda| - c^{1/2}\|_2 = 0. \end{aligned}$$

Since we have the inequality

$$\begin{aligned} &\|\rho(x) - c^{1/2} u_\lambda^* x u_\lambda c^{1/2}\|_2 \\ &\quad \leq \|\rho(x) - a_\lambda^* x a_\lambda\|_2 + \|a_\lambda^* x a_\lambda - c^{1/2} u_\lambda^* x u_\lambda c^{1/2}\|_2 \end{aligned}$$

for every $x \in M$, it is sufficient to show the relation $\lim \|a_\lambda^* x a_\lambda - c^{1/2} u_\lambda^* x u_\lambda c^{1/2}\|_2 = 0$. For every $x \in M$, we have the relation

$$\begin{aligned} &\|a_\lambda^* x a_\lambda - c^{1/2} u_\lambda^* x u_\lambda c^{1/2}\|_2 \\ &\quad = \|a_\lambda^* x a_\lambda - a_\lambda^* x u_\lambda c^{1/2} + a_\lambda^* x u_\lambda c^{1/2} - c^{1/2} u_\lambda^* x u_\lambda c^{1/2}\|_2 \\ &\quad \leq \|a_\lambda^* x (a_\lambda - u_\lambda c^{1/2})\|_2 + \|(a_\lambda^* - c^{1/2} u_\lambda^*) x u_\lambda c^{1/2}\|_2 \\ &\quad \leq \|a_\lambda\| \cdot \|x\| \cdot \|a_\lambda - u_\lambda c^{1/2}\|_2 + \|u_\lambda c^{1/2}\| \cdot \|x\| \cdot \|a_\lambda - u_\lambda c^{1/2}\|_2. \end{aligned}$$

Since $\{a_\lambda\}$ is a bounded net, by using the property $\lim \|a_\lambda - u_\lambda c^{1/2}\|_2 = 0$ we have the relation

$$\lim \|a_\lambda^* x a_\lambda - c^{1/2} u_\lambda^* x u_\lambda c^{1/2}\|_2 = 0$$

for every $x \in M$. Thus, the positive linear map ρ is approximately inner with respect to the net $\{u_\lambda c^{1/2}\}$ and also $\{v_\lambda c^{1/2}\}$. Q.E.D.

THEOREM 4. *Let* M *be a* σ*-finite, finite on Neumann algebra with a faithful, normalized normal trace* Tr. *Let* ρ *be a positive map of* M *into itself such that* $\rho(1) = c$ *and* $f = \mathrm{supp}(c)$. *Let* $\{a_\lambda\}$ *be a bounded net in* M. *We suppose that there exists a projection* e *in* M *such that* $a_\lambda = e a_\lambda$ *for every* λ, $\mathrm{Tr}(e) = \mathrm{Tr}(f)$ *and* ρ *is approximately inner with respect to* $\{a_\lambda\}$. *Then there exists a* **-homomorphism* π *of* eMe *into* fMf *such that* $\rho(x) = c^{1/2}\pi(x)c^{1/2}$ *for every* $x \in M$.

PROOF. Put

$$\pi_n(x) = (c + 1/n)^{-1/2} \rho(x) (c + 1/n)^{-1/2}$$

for every $x \in M$. Then, by Lemma 2, the sequence $\{\pi_n(x)\}$ converges in the strong topology for every $x \in M$. Thus, we define a positive linear map π by

$$\pi(x) = \text{ the strong-limit of } \pi_n(x).$$

Then π is a completely positive map, $\pi(1) = f$ and $\rho(x) = c^{1/2}\pi(x)c^{1/2}$ for every $x \in M$. Hence, we shall show that π is a *-homomorphism of eMe into fMf. For showing it, we have the following estimates: Since the sequence $\{c^{1/2}(c+1/n)^{-1/2}\}$ is bounded and converges to f in the strong topology, $\lim \|c^{1/2}(c+1/n)^{-1/2} - f\|_2 = 0$. Thus, for an arbitrary positive number ε, there exists a natural number N' such that $\|c^{1/2}(c+1/n)^{-1/2} - f\|_2 < \varepsilon/6$ for every $n > N'$. By using the above inequality, we have the following relation for every $y \in S$ where S is the unit ball of M; for every $n \geq N'$,

$$\begin{aligned}
&\|(c+1/n)^{-1/2}c^{1/2}yc^{1/2}(c+1/n)^{-1/2} - fyf\|_2 \\
&\quad \leq \|(c+1/n)^{-1/2}c^{1/2}yc^{1/2}(c+1/n)^{-1/2} - (c+1/n)^{-1/2}c^{1/2}yf\|_2 \\
&\qquad + \|(c+1/n)^{-1/2}c^{1/2}yf - fyf\|_2 \\
&\quad = \|(c+1/n)^{-1/2}c^{1/2}y(c^{1/2}(c+1/n)^{-1/2} - f)\|_2 \\
&\qquad + \|(c+1/n)^{-1/2}(c^{1/2} - f)yf\|_2 \\
&\quad \leq \|(c+1/n)^{-1/2}c^{1/2}y\| \cdot \|c^{1/2}(c+1/n)^{-1/2} - f\|_2 \\
&\qquad + \|y\| \cdot \|(c+1/n)^{-1/2}c^{1/2} - f\|_2 \\
&\quad \leq 2\|c^{1/2}(c+1/n)^{-1/2} - f\|_2 < \varepsilon/3.
\end{aligned}$$

From the above inequality, if we take an arbitrary element $x \in S$ and $n \geq N'$, then we have the inequality

$$\begin{aligned}
&\|(c+1/n)^{-1/2}c^{1/2}u_\lambda^* x u_\lambda c^{1/2}(c+1/n)^{-1/2} - fu_\lambda^* x u_\lambda f\|_2 \\
(*) \qquad &\quad = \|(c+1/n)^{-1/2}c^{1/2}v_\lambda^* x v_\lambda c^{1/2}(c+1/n)^{-1/2} - v_\lambda^* x v_\lambda\|_2 \\
&\quad < \varepsilon/3
\end{aligned}$$

for every λ. Furthermore, since the sequence $\{\pi_n(x)\}$ is bounded and converges to $\pi(x)$ in the strong topology, $\lim_{n\to\infty} \|\pi_n(x) - \pi(x)\|_2 = 0$. Thus, there exists a natural number N'' such that, for every $n \geq N''$,

$$(**) \qquad \|\pi_n(x) - \pi(x)\|_2 < \varepsilon/3.$$

Let $N = \max\{N', N''\}$. Now, we choose a fixed natural number n with $n \geq N$. Then, since

$$\lim_\lambda \|(c+1/n)^{-1/2}c^{1/2}u_\lambda^* x u_\lambda c^{1/2}(c+1/n)^{-1/2} - \pi_n(x)\|_2 = 0,$$

there exists an index λ_0 for the net $\{a_\lambda\}$ such that, for every $\lambda \geq \lambda_0$,

$$(***) \qquad \|(c+1/n)^{-1/2}c^{1/2}u_\lambda^* x u_\lambda c^{1/2}(c+1/n)^{-1/2} - \pi_n(x)\|_2 < \varepsilon/3.$$

Considering relations $(*)$, $(**)$ and $(***)$, we then have the relation

$$\|\pi(x) - fu_\lambda^* x u_\lambda f\|_2 = \|\pi(x) - v_\lambda^* x v_\lambda\|_2 < \varepsilon$$

for every $\lambda \geq \lambda_0$. Thus,

$$\lim_\lambda \|\pi(x) - fu_\lambda^* x u_\lambda f\|_2 = \lim_\lambda \|\pi(x) - v_\lambda^* x v_\lambda\|_2 = 0$$

for every $x \in M$. Hence, π is approximately inner with respect to the net $\{u_\lambda f\}$ $(= \{v_\lambda\})$ and $\pi(1) = f$. Furthermore, since $(1-e)v_\lambda = 0$ for every λ, $\pi(1-e) = 0$. Therefore, by Theorem 3 and Remark 4 in [**5**], π is a *-homomorphism of eMe into fMf. Thus, we have the complete proof of Theorem 4. Q.E.D.

COROLLARY 5. *Let ρ be a positive linear map of a σ-finite, finite von Neumann algebra M into itself such that the support projection of $\rho(1)$ is 1. If ρ is approximately inner with respect to a bounded net in M, then π in Theorem 4 is a *-homomorphism of M into itself.*

REMARK 6. We obtained Theorem 3 in [**5**] without the assumption of the boundedness for the net $\{a_\lambda\}$, but we assumed the boundedness in this paper. Can this restriction be removed?

ADDED IN PROOF (SEPTEMBER 21, 1986). We asked the question in Remark 6 whether the restriction of the boundedness for the net $\{a_\lambda\}$ can be removed. After we submitted this paper to this proceedings, the second author (H. Takemoto) showed that this restriction can be removed. Thus we can show Theorem 4 in our paper without the assumption of the boundedness for $\{a_\lambda\}$.

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REFERENCES

1. M. D. Choi, *A Schwartz inequality for positive linear maps on C^*-algebras*, Illinois J. Math. **18** (1974), 565–574.
2. U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. **37** (1975), 271–283.
3. G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Math. Soc. Monographs, vol. 14, Academic Press, London, 1979.
4. W. F. Stinespring, *Positive maps on C^*-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211–216.
5. H. Takemoto, *Approximately innerness of positive linear maps of finite von Neumann algebras*, Proc. Amer. Math. Soc. **94** (1985), 463–466.
6. M. Takesaki, *Theory of operator algebras*. I, Springer-Verlag, Berlin and New York, 1979.
7. J. Tomiyama, *Complete positivity in operator algebras*, Lecture Note No. 4, RIMS Kyoto Univ., 1978. (Japanese)

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