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GENERALIZATIONS OF THE SIDON-TELYAKOVSKII THEOREM

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ABSTRACT. The well-known Sidon-Telyakovskii integrability condition is considerably lightened as follows:

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta c(k)|^p}{A_k^p}=O(1), \qquad n\to\infty,$$

where $\{c(n)\}$ is a certain null-sequence and $1<p\leq 2$. It is proved that $\sum_{n=1}^{\infty} n^{p-1}|\Delta c(n)|^p\rho^p(n)<\infty$ is also a sufficient integrability condition provided $\sum_{n=1}^{\infty}(1/n\rho(n))<\infty$, where $\{\rho(n)\}$ is an increasing sequence of positive numbers.

1. Introduction. S. A. Telyakovskii [**1**] found a succinct equivalent form of the Sidon [**2**] integrability condition of cosine series. In [**1**] the following class S of real null sequences is defined. A real null sequence $\{a_n\}$ belongs to S if there exists a monotone sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty}A_n<\infty$ and

$$|\Delta a_n|\leq A_n \quad \text{for all } n. \tag{1.1}$$

The Sidon-Telyakovskii theorem states that if $\{a_n\}\in S$ then the series $a_0/2+\sum_{n=1}^{\infty}a_n\cos nx$ is the Fourier series of its sum f, and $\|S_n(f)-f\|=o(1)$, $n\to\infty$, is equivalent to $a_n \lg n=o(1)$, $n\to\infty$, where $S_n(f)=S_n(f,x)=a_0/2+\sum_{k=1}^{n}a_k\cos kx$ and $\|\cdot\|$ denotes $L^1(0,\pi)$-norm (a denotation to be used throughout the paper).

The purpose of this paper is to weaken (1.1) to the condition

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta c(k)|^p}{A_k^p}=O(1), \qquad n\to\infty, \tag{1.2}$$

where $p>1$, $\{A_n\}$ is a monotone sequence such that $\sum_{n=1}^{\infty}A_n<\infty$, and $\{c(n)\}$ is a null sequence of complex numbers. To control the sine part of the complex trigonometric series $\sum_{|n|<\infty}c(n)e^{int}$, $t\in T=R/2\pi Z$, a technical condition is needed. A complex null sequence $\{c(n)\}$ satisfying $\sum_{n=1}^{\infty}|\Delta(c(n)-c(-n))|\lg n<\infty$, is called weakly even. It is plain that if $\{c(n)\}$ is an even sequence then it is weakly even.

The partial sums of the complex trigonometric transform $\sum_{|n|<\infty}c(n)e^{int}$ will be denoted by $S_n(c)=S_n(c,t)=\sum_{|k|\leq n}c(k)e^{ikt}$. If a trigonometric transform is the Fourier transform of some $f\in L^1$, we shall write $c(n)=\hat{f}(n)$ for all n, and $S_n(c,t)=S_n(f,t)$.

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The following definition will be useful in the formulation of our main result.

DEFINITION 1.1. A weakly even null sequence $\{c(n)\}$ of complex numbers belongs to the class S_p^* if for some $1 < p \leq 2$ and some monotone sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$, the condition (1.2) holds.

2. Generalizations of the Sidon-Telyakovskii theorem. The main result in this paper is the following theorem that generalizes the Sidon-Telyakovskii result.

THEOREM 2.1. *Let* $\{c(n)\} \in S_p^*$. *Then*

(i) *for* $t \neq 0$, $\lim_{n\to\infty} S_n(c,t) = f(t)$ *exists*;

(ii) $f \in L^1(T)$;

(iii) $\|S_n(f) - f\| = o(1)$, $n \to \infty$, *is equivalent to* $\hat{f}(n)\lg|n| = o(1)$, $|n| \to \infty$.

PROOF. To show that the condition (1.2) implies (i) it suffices to show that $\{c(n)\}$ is bounded variation. Indeed,

$$\sum_{k=1}^{n} |\Delta c(k)| = \sum_{k-1}^{n-1} [\Delta A_k] \sum_{j=1}^{k} \frac{|\Delta c(j)|}{A_j} + A_n \sum_{j=1}^{n} \frac{|\Delta c(j)|}{A_j}$$
$$\leq \sum_{k=1}^{n-1} k[\Delta A_k] \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta c(j)|^p}{A_j^p} \right)^{1/p} + nA_n \left(\frac{1}{n} \sum_{j=1}^{n} \frac{|\Delta c(j)|^p}{A_j^p} \right)^{1/p}.$$

Hence $\{c(n)\}$ is of bounded variation and for $t \neq 0$, $\lim_{n\to\infty} S_n(c,t)$ exists. We denote this limit by $f(t)$.

For the proof that $f \in L^1(T)$ we need the complex form of the so-called modified trigonometric sums introduced by J. W. Garrett and Č. V. Stanojević [**3**, **4**]. Let $D_n(t) = \sin(n+1/2)t/(\sin t/2)$ denote the Dirichlet kernel in the complex case and let $E_n(t) = \sum_{k=0}^{n} e^{ikt}$. Then

$$S_n(c,t) - (c(n)E_n(t) + c(-n)E_{-n}(t))$$
$$= g_n(c,t) = \sum_{k=1}^{n-1} (\Delta(c(-k) - c(k)))(E_{-k}(t) - 1) - c(-n) + \sum_{k=0}^{n-1} \Delta c(k) D_k(t).$$

From (i) it follows that for $t \neq 0$

$$f(t) - g_n(c,t) = \sum_{k=n}^{\infty} \Delta c(k) D_k(t) + \sum_{k=n}^{\infty} (\Delta(c(-k) - c(k)))E_{-k}(t).$$

From the last identity we have the estimate

$$\|f - g_n(c)\| \leq \int_T \left| \sum_{k=n}^{\infty} \Delta c(k) D_k(t) \right| dt + B_1 \sum_{k=n}^{\infty} |\Delta(c(-k) - c(k))| \lg n,$$

where B_1 is an absolute constant. Since $\{c(n)\}$ is weakly even, the second term on the right-hand side of the above inequality is also $o(1)$, as $n \to \infty$. Thus

$$\|f - g_n(c)\| \leq B_2 \int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta c(k) D_k(t) \right| dt + o(1), \qquad n \to \infty,$$

where B_2 is an absolute constant. It remains to show that the integral on the right-hand side of the last inequality vanishes as $n \to \infty$.

For $t \neq 0$ consider the identity

$$\sum_{k=n}^{\infty} \Delta c(k) D_k(t) = \sum_{k=n-1}^{\infty} \Delta A_k \sum_{j=1}^{k} \frac{\Delta c(j)}{A_j} D_j(t) - A_n \sum_{j=1}^{n-1} \frac{\Delta c(j)}{A_j} D_j(t).$$

Then

$$\int_0^\pi \left| \sum_{k=n}^{\infty} \Delta c(k) D_k(t) \right| dt \leq \sum_{k=n}^{\infty} \Delta A_k \int_0^\pi \left| \sum_{j=1}^{k} \frac{\Delta c(j)}{A_j} D_j(t) \right| dt + A_n \int_0^\pi \left| \sum_{j=1}^{n} \frac{\Delta c(j)}{A_j} D_j(t) \right| dt.$$

Both integrals on the right-hand side of the above inequality can be estimated in the same way. Namely,

$$\int_0^\pi \left| \sum_{j=1}^{N} \frac{\Delta c(j)}{A_j} D_j(t) \right| dt = \int_0^{\pi/N} \left| \sum_{j=1}^{N} \frac{\Delta c(j)}{A_j} D_j(t) \right| dt + \int_{\pi/N}^{\pi} \left| \sum_{j=1}^{N} \frac{\Delta c(j)}{A_j} D_j(t) \right| dt = I_N + J_N.$$

Recalling the uniform estimate of the Dirichlet kernel we have

$$I_N \leq B_3 \sum_{j=1}^{N} \frac{|\Delta c(j)|}{A_j} \leq B_3 N \left(\frac{1}{N} \sum_{j=1}^{N} \frac{|\Delta c(j)|^p}{A_j^p} \right)^{1/p},$$

where B_3 is an absolute constant. To estimate the second integral

$$J_N = \int_{\pi/N}^{\pi} \left| \sum_{j=1}^{N} \frac{\Delta c(j)}{A_j} D_j(t) \right| dt = \int_{\pi/N}^{\pi} \frac{1}{\sin t/2} \left| \sum_{j=1}^{N} \frac{\Delta c(j)}{A_j} \sin(j+1/2)t \right| dt,$$

we shall first apply the Hölder inequality, where $1/p + 1/q = 1$,

$$J_N \leq \left[\int_{\pi/N}^{\pi} \left(\frac{1}{\sin t/2} \right)^p dt \right]^{1/p} \left[\int_0^\pi \left| \sum_{j=1}^{N} \frac{\Delta c(j)}{A_j} \sin(j+1/2)t \right|^q dt \right]^{1/q}$$

followed by the Hausdorff-Young inequality

$$J_N \leq B_4 N^{1/q} \left[\sum_{j=1}^{N} \frac{|\Delta c(j)|^p}{A_j^p} \right]^{1/p}.$$

Finally,

$$J_N \leq B_4 N \left(\frac{1}{N} \sum_{j=1}^{N} \frac{|\Delta c(j)|^p}{A_j^p} \right)^{1/p},$$

where B_4 is an absolute constant. Thus

$$\int_0^\pi \left| \sum_{k=n}^{\infty} \Delta c(k) D_k(t) \right| dt \leq B_5 \sum_{k=n}^{\infty} k \Delta A_k + B_6 n A_n,$$

where B_5 and B_6 are absolute constants. Since $\sum_{n=1}^{\infty} A_n < \infty$, both terms on the right-hand side of the above inequality are $o(1)$, as $n \to \infty$. Therefore

$$\|f - g_n(c)\| = o(1), \qquad n \to \infty,$$

and since g_n is a polynomial, it follows that f is integrable. This completes the proof of (ii).

The proof of (iii) follows from

$$\left| \|f - S_n(f)\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| \right| \leq \|f - g_n(c)\| = o(1), \qquad n \to \infty,$$

and from the fact in [**5**] that

$$\|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| = o(1), \qquad n \to \infty,$$

is equivalent to

$$\hat{f}(n) \lg |n| = o(1), \qquad |n| \to \infty.$$

In proving (1) we used condition (1.2) as a generalized comparison test for the convergence of the series of positive terms. This indicates that the improvements of Theorem 2.1 should be sought through refinement of our test.The other possibility is to look at condition (1.2) as some kind of regularity and speed condition for $\{c(n)\}$. These remarks become more transparent if we look at corollaries of Theorem 2.1, namely the Sidon-Telyakovskii theorem for complex weakly even case.

COROLLARY 2.1. *Let $\{c(n)\}$ be a weakly even complex null sequence satisfying condition* (1.1). *Then,*

(i) *for $t \neq 0$, $\lim_{n\to\infty} S_n(c,t) = f(t)$ exists*;
(ii) $f \in L^1(T)$;
(iii) $\|S_n(f) - f\| = o(1)$, $n \to \infty$, *is equivalent to* $\hat{f}(n)\lg|n| = o(1), |n| \to \infty$.

Both condition (1.1) and its weaker form (1.2) are not intrinsic conditions, for they depend on a monotone sequence $\{A_n\}$ used as a comparison tool. It would be of considerable interest to replace (1.2) by an intrinsic condition, namely a condition involving only $\{c(n)\}$.

3. Another integrability theorem. The modified sums g_n can also be used to obtain integrability conditions in terms of convergence of certain series.

G. A. Fomin [**6**] observed that if the null sequence of real numbers $\{a_n\}$ satisfies

$$\sum_{n=1}^{\infty} \left[\frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n} \right]^{1/p} < \infty \quad \text{for some } 1 < p \leq 2, \tag{3.1}$$

then the cosine series with coefficients $\{a_n\}$ is the Fourier series of its sum, and it converges in $L^1(0, \pi)$-norm if and only if

$$a_n \lg n = o(1), \qquad n \to \infty.$$

In proving his result Fomin showed that (3.1) implies

$$\sum_{n=1}^{\infty} n^{p-1} |\Delta a_n|^p < \infty \quad \text{for some } 1 < p \leq 2. \tag{3.2}$$

It can be shown that (3.2)is effectively weaker than (3.1) and that (3.2) imples that $\{a_n\}$ is of bounded variation.

The above remarks bring us to the following questions:

(i) Is the condition (3.2) sufficient for the integrability of the corresponding cosine series?

(ii) Is there a slightly stronger form of (3.2), different from (3.1), that would imply the integrability of the corresponding cosine series?

It is probable that the answer to question (i) is negative. Concerning (ii) the following theorem gives an affirmative answer. It also provides an insight into the degree of strengthening of (3.2) so that it becomes a sufficient integrability condition.

THEOREM 3.1. *Let $\{c(n)\}$ be a weakly even null sequence of complex numbers and let $\{\rho(n)\}$ be an increasing sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} (1/n\rho(n)) < \infty.$$

If for some $1 < p \leq 2$

$$\sum_{n=1}^{\infty} n^{p-1} |\Delta c(n)|^p \rho^p(n) < \infty, \tag{3.3}$$

then

(i) *for $t \neq 0$, $\lim_{n\to\infty} S_n(c,t) = f(t)$ exists;*

(ii) $f \in L^1(T)$;

(iii) $\|S_n(f) - f\| = o(1), n \to \infty$, *is equivalent to* $\hat{f}(n)\lg|n| = o(1), |n| \to \infty$.

PROOF. From (3.3) follows (3.2), and from (3.2) it follows that $\{c(n)\}$ is of bounded variation. Hence, for $t \neq 0$, $\lim_{n\to\infty} S_n(c,t) = f(t)$ in T. Again, using the modified sums g_n, we obtain the estimate

$$\|g_n(c) - f\| \leq C \sum_{k=n}^{\infty} \frac{1}{k\rho(k)} \sum_{j=1}^{k} j^{p-1} |\Delta c(j)|^p \rho^p(j) + o(1), \qquad n \to \infty,$$

where C is an absolute constant. The proof that the convergence in $L^1(T)$-norm is equivalent to $\hat{f}(n)\lg|n| = o(1)$, $|n| \to \infty$, goes along the standard lines.

It seems that by using Hölder-Hausdorff-Young techniques, condition (3.3) cannot be weakened if we are to have a sufficient condition for integrability.

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