PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 4, December 1987

A SHORT PROOF OF A THEOREM ON COMPOSITIONS OF D.C. MAPPINGS

LIBOR VESELÝ

(Communicated by R. Daniel Mauldin)

ABSTRACT. In this note, there is presented an essentially shorter proof of the theorem that a composition of d.c. mappings is locally d.c., proved for finite dimensions by P. Hartman in 1959.

A mapping $F = (F_1, F_2, \ldots, F_n)$ from an open convex set $A \subset \mathbf{R}^k$ into $\mathbf{R}^n$ is called d.c. if each of its components F_j is representable as a difference of two convex functions on A. P. Hartman proved that the composition of d.c. mappings is locally d.c. in the sense that every point in A has a convex open neighborhood, on which the composite mapping is d.c. (see [**1**, (II)]).

In this paper, we give a simpler and much shorter proof of this fact. To prove a rather more general version of Hartman's theorem, we state the following definition. (All linear spaces in this note are real linear spaces. The continuous dual of a normed linear space X is denoted by X^*.)

DEFINITION. Let X, Y be normed linear spaces and let $A \subset X$ be a nonempty open convex set. A mapping $F: A \to Y$ is said to be d.c. on A iff F is continuous and there exists a continuous convex function f on A such that for any y^* from a unit sphere in Y^* the function $y^* \circ F + f$ is convex on A. (We shall say that F is d.c. on A with convex function f.)

It is easy to see that this definition is equivalent to the one written above, in case $X = \mathbf{R}^k$, $Y = \mathbf{R}^n$.

For a continuous convex function f on an open convex subset A of a normed linear space X and for $x_0 \in A$ we denote $\partial f(x_0) = \{x^* \in X^*: f(x) - f(x_0) \geq \langle x - x_0, x^* \rangle$ for any $x \in A\}$. It is a well-known fact that this set (called a subdifferential of f at x_0) is always nonempty (see [**2**, §43]).

THEOREM. *Let X, Y be Banach spaces and let Z be a normed linear space. Let $A \subset X$, $B \subset Y$ be nonempty open convex sets. Let a mapping $F: A \to Y$ be d.c. on A and $F(A) \subset B$. Let $G: B \to Z$ be d.c. and locally Lipschitz on B. Then $G \circ F$ is locally d.c. on A.*

Note that in case $Z = \mathbf{R}^m$, it is possible to omit the assumption of local Lipschitz property of G as it is fulfilled automatically.

PROOF. Let F (G, respectively) be d.c. on A (on B, resp.) with convex function f (g, resp.). Let $a_0 \in A$ be an arbitrary point. Let $V \subset B$ be an open convex neighborhood of $F(a_0)$ such that both G, g are Lipschitz on V (with constants L_G, L_g) and let $U \subset A$ be an open convex neighborhood of a_0 with $F(U) \subset V$.

Received by the editors August 19, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26B35, 47H99; Secondary 26B25, 26B40.

Denote $M = L_G + L_g$. We shall show that $G \circ F$ is d.c. on U with convex function $g \circ F + Mf$. Let $x_0 \in U$.

(i) Let $x^* \in \partial f(x_0), y^* \in \partial g(F(x_0))$ and let $y_0^* \in Y^*$ be such that $\|y_0^*\| = 1$ and $y^* = \|y^*\| y_0^*$. Let $u^* \in \partial(y_0^* \circ F + f)(x_0)$. Then any $x \in U$ satisfies

$$\begin{aligned} g(F(x)) &+ Mf(x) - [g(F(x_0)) + Mf(x_0)] \\ &\geq \langle F(x) - F(x_0), y^* \rangle + M[f(x) - f(x_0)] \\ &= \|y^*\|[\langle F(x) - F(x_0), y_0^* \rangle + f(x) - f(x_0)] + (M - \|y^*\|)[f(x) - f(x_0)] \\ &\geq \|y^*\| \langle x - x_0, u^* \rangle + (M - \|y^*\|)\langle x - x_0, x^* \rangle, \end{aligned}$$

since $\|y^*\| \leq L_g \leq M$. This means that the function $g \circ F + Mf$ has a continuous support at an arbitrary point $x_0 \in U$. Hence this function is continuous and convex on U (see [**2**, Theorem 43C]).

(ii) Let $z^* \in Z^*$ satisfy $\|z^*\| = 1$, let $x^* \in \partial f(x_0), v^* \in \partial(z^* \circ G + g)(F(x_0))$, and let $v_0^* \in Y^*$ be such that $\|v_0^*\| = 1$ and $v^* = \|v^*\| v_0$. Let $w^* \in \partial(v_0^* \circ F + f)(x_0)$. Then $\|v^*\| \leq M$ and the following inequalities hold for any $x \in U$:

$$\begin{aligned} \langle G(f(x)), z^* \rangle &+ g(F(x)) + Mf(x) - [\langle G(F(x_0)), z^* \rangle + g(F(x_0)) + Mf(x_0)] \\ &\geq \langle F(x) - F(x_0), v^* \rangle + \|v^*\|[f(x) - f(x_0)] + (M - \|v^*\|)[f(x) - f(x_0)] \\ &\geq \|v^*\| \langle x - x_0, w^* \rangle + (M - \|v^*\|)\langle x - x_0, x^* \rangle. \end{aligned}$$

The function $z^* \circ G \circ F + g \circ F + Mf$ is continuous and convex on U by the same argument as in (i). The theorem is proved.

References

1. P. Hartman, *On functions representable as a difference of convex functions*, Pacific J. Math. **9** (1959), 707–713.
2. A. W. Roberts and D. E. Varberg, *Convex functions*, Academic Press, New York and London, 1973.

Universita Karlova, Matematicko-Fysikální Fakulta, KMA, Sokolovská 83, 18600 Praha 8, Czechoslovakia