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MORE QUASI-REFLEXIVE SUBSPACES

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ABSTRACT. It is shown that nonreflexive Banach spaces with a separable dual and the boundedly complete skipped blocking property have quasi-reflexive subspaces. In particular, Bourgain's somewhat reflexive $\mathcal{L}_\infty$-spaces and Polish Banach spaces are somewhat quasi-reflexive.

The author in [**1**] has shown that most collections of Banach spaces which are known to contain a reflexive subspace also contain a quasi-reflexive subspace. (Of course, the words "infinite dimensional" are needed to make this correct. Banach spaces are assumed to be infinite dimensional unless otherwise stated.) Indeed, the following question is still open:

(1) Does each nonreflexive Banach space contain c_0, l_1 or a quasi-reflexive subspace?

An affirmative answer to (1) wôuld imply an affirmative answer to the well-known question:

(2) Does each Banach space contain c_0, l_1 or a reflexive subspace?

In [**3**] (or see [**2**]), Bourgain and Delbaen construct a collection of somewhat reflexive $\mathcal{L}_\infty$-spaces. It was suggested to the author that these spaces might yield a negative answer to question (1). This note shows that these $\mathcal{L}_\infty$-spaces contain quasi-reflexive spaces, and hence (1) is still open.

In [**4**] (or see [**7**]), Edgar and Wheeler show that each Polish Banach space contains a reflexive subspace. (A Banach space is *Polish* if its unit ball with the weak topology is a Polish topological space, i.e. homeomorphic to a separable complete metric space). Our Theorem 3 implies that each nonreflexive Polish space has quasi-reflexive subspaces. Indeed, X being Polish is equivalent to X^* being separable and X has PCP [**4**] (or see [**7**]); while X has PCP is equivalent to X having the boundedly complete skipped blocking property [**5**] (or see [**7**]).

We mention a short footnote to [**1**]. In [**1**] it was shown that most positive answers to question (2) also have a positive answer to question (1). The only exception noted in [**1**] was the space X^* when X^{**}/X is separable. However, Valdivia [**8**] had already shown that $X = R \oplus Y$ where Y^{**} is separable and R is reflexive. Hence $X^* = R^* \oplus Y^*$ and X^* has quasi-reflexive subspaces by Theorem 9 of [**1**].

Our notation is standard and follows that of [**6**]. Also this paper is a sequel to [**1**], where some details are more carefully explained.

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For any basic sequence (e_n), the space $(e_n)^{\text{LIM}}$ is the collection of scalar sequences (a_n) so that

$$\|(a_n)\| = \sup \left\| \sum_1^k a_n e_n \right\| < \infty.$$

For $(a_n) \in (e_n)^{\text{LIM}}$, we will routinely identify (a_n) with the formal sum $\sum a_n e_n$. Let (f_n) be the coefficient functionals to (e_n). We have $[e_n]^*$ is isomorphic to $(f_n)^{\text{LIM}}$. The basis (e_n) is *boundedly complete* (resp. *k-boundedly complete, shrinking*) if $[e_n] = (e_n)^{\text{LIM}}$ (resp. $\dim((e_n)^{\text{LIM}}/[e_n]) = k$, (f_n) is boundedly complete).

A Banach space X is *quasi-reflexive* (of order k) if $\dim X^{**}/X$ is finite (is k-dimensional). We will assume quasi-reflexive implies nonreflexive.

A *boundedly complete SBD* (Skipped Blocking Decomposition) for a space X is a sequence of finite dimensional subspaces (X_n) with dense linear span with the properties:

(1) $X_i \cap [X_j]_{j \neq 1} = \{0\}$.

(2) If $(M(k))$ and $(N(k))$ are sequences of positive integers so that $M(k) < N(k) + 1 < M(k+1)$, then the sequence $Y_k = [X_i]_{i=M(k)}^{N(k)}$ is a boundedly complete FDD for its closed linear span.

If (e_n) is a basic sequence and $S_n = [e_i]_{i=n}^{n}$ is a boundedly complete SBD, then we will say (e_n) is *skipped blocked boundedly complete.*

A space X is said to have the *boundedly complete SBP* if it has a boundedly complete SBD.

A sequence (e_n) will be called *seminormalized* if $0 < \liminf \|e_n\| \leq \limsup \|e_n\| < \infty$.

LEMMA 1. *A seminormalized basis (e_n), which is skipped blocked boundedly complete, but $\sum e_n \in (e_n)^{\text{LIM}}$ (so it is not boundedly complete) satisfies*

$$(e_n)^{\text{LIM}} = \left[(e_n) \cup \left\{\sum e_n\right\}\right]$$

and so (e_n) is 1-boundedly complete.

PROOF. Let $\sum a_n e_n \in (e_n)^{\text{LIM}}$. First we will show that if there is a sequence $N(k) + 1 < N(k+1)$ with $N(0) = 0$ so that $a_{N(k)} = 0$, then $\sum a_n e_n \in [e_n]$. Let $b_k = \sum_{N(k-1)+1}^{N(k)-1} a_n e_n$; then (b_k) is a skipped block basic sequence, and is hence boundedly complete. Thus $\sum a_n e_n = \sum b_k \in [b_k] \subset [e_n]$.

Suppose there is a similar sequence $N(k) + 1 < N(k+1)$ with $N(0) = 0$ so that $\sum |a_{N(k)}| < \infty$. Obviously $x = \sum a_{N(k)} e_{N(k)} \in [e_n]$ and by the above $\sum a_n e_n - x \in [e_n]$. Therefore, if zero is a cluster point of the sequence (a_n) then $\sum a_n e_n \in [e_n]$.

In general, let A be a cluster point of (a_n) for $\sum a_n e_n \in (e_n)^{\text{LIM}}$. We have $\sum a_n e_n - A \sum e_n \in [e_n]$, which proves the lemma and shows $\lim a_n = A$. □

LEMMA 2. *Let X have a boundedly complete SBD (X_i) and suppose X has a seminormalized basic sequence (e_n) so that $\sum e_n \in (e_n)^{\text{LIM}}$. Then there is an increasing sequence $(N(k))$ with $N(0) = 0$ so that if*

$$b_k = \sum_{N(k-1)+1}^{N(k)} e_n,$$

then $(b_k)^{\text{LIM}} = [(b_k) \cup \{\sum b_k\}]$.

PROOF. Since (X_i) has dense linear span we may assume (passing to an equivalent basic sequence if necessary) there is an increasing sequence $(U(n))$ so that $e_n \in [X_i]_1^{U(n)}$.

Let P_i be the natural projection onto X_i so that $P_iP_j = 0$ for $i \neq j$ and let $Q_i = \sum_{j=1}^{i} P_j$. Since the sequence $(\sum_1^m e_n)_m$ is bounded, there is a subsequence $m(k)$ so that for a fixed i, $(Q_i(\sum_1^{m(k)} e_n))_k$ converges. By passing to subsequences and diagonalizing we obtain a subsequence $(M(k))$ with $M(0) = 0$ so that

$$\left\| Q_i \left(\sum_{M(i)+1}^{M(i+1)} e_n \right) \right\| < \delta/2^i,$$

where δ is small compared to the basis constant of (e_n). Thus $(\sum_{M(k-1)+1}^{M(k)} e_n)_k$ is equivalent to a basic sequence (d_k) with the property that

$$d_k \in [X_i]_k^{U(k)},$$

and so we can assume $Q_i e_{i+1} = 0$.

Now inductively pick $N(k), M(k)$ so that $N(0) = 0$, $N(i) = 1, (e_n)_{N(k-1)+1}^{N(k)} \subset [X_i]_1^{M(k)}$, and $N(k+1) = M(k) + 2$. Now if $b_k = \sum_{N(k-1)+1}^{N(k)} e_n$, then $b_k \in [X_i]_{N(k-1)+1}^{N(k+1)-1}$ and hence (b_k) is a skipped block boundedly complete basic sequence. Now (b_k) is bounded since $\sum e_n \in (e_n)^{\mathrm{LIM}}$ and thus (b_k) is seminormalized since (e_n) is bounded away from zero. Furthermore, since $\sum b_k = \sum e_n$, $(b_k)^{\mathrm{LIM}} = [(b_k) \cup \{\sum b_k\}]$ by Lemma 1. □

THEOREM 3. *A nonreflexive space with a separable dual and the boundedly complete SBP has a quasi-reflexive subspace.*

PROOF. Let X be such a space. Since X^* is separable, l_1 is not a subspace of X. Thus by [**6**, p. 101] each $x^{**} \in X^{**} \setminus X$ is the $\sigma(X^{**}, X^*)$-limit of some sequence $(x_n) \subset X$. The principle of uniform boundedness implies (x_n) is a bounded sequence. As in the proof of Theorem 8 of [**1**, p. 181] we may assume (x_n) is basic, and (e_n), defined by $e_1 = x_1$ and $e_n = x_n - x_{n-1}$, is also basic. Note that we may also assume that (e_n) is seminormalized.

Thus $\sum e_n \in (e_n)^{\mathrm{LIM}}$ and $(\sum_1^m e_n)$ converges to x^{**} in $\sigma(X^{**}, X^*)$. By Lemma 2, we may assume $(e_n)^{\mathrm{LIM}} = [(e_n) \cup \{\sum e_n\}]$. Now $[e_n]^*$ is separable, since it is a quotient of X^*. Hence by Lemma 3 of [**1**, p. 178], we may assume (e_n) is shrinking. Thus $[e_n]^{**} = [(e_n) \cup \{\sum e_n\}]$, so (e_n) is quasi-reflexive of order one and if $S\colon [e_n] \to X$, then $S^{**}(\sum e_n) = x^{**}$. □

REMARK. Since having a separable dual and the boundedly complete SBP [**2**, p. 17] are hereditary, each nonreflexive subspace of such a space has a quasi-reflexive subspace. In [**1**] such spaces were called *somewhat quasi-reflexive*.

COROLLARY 4. *The somewhat reflexive $\mathcal{L}_\infty$-spaces in* [**2**, **3**] *are also somewhat quasi-reflexive.*

PROOF. In [**2**, **3**] it is shown that these spaces have the boundedly complete SBP and their duals are isomorphic to l_1. □

COROLLARY 5. *A Polish Banach space is somewhat quasi-reflexive.*

PROOF. Each nonreflexive subspace of a Polish space satisfies the hypothesis of Theorem 3 (see [**7**]) hence has a quasi-reflexive subspace.

REMARKS. 1. If X is one of the $\mathcal{L}_\infty$-somewhat reflexive spaces of [**3**], then X^{**} is isomorphic to l_∞. Thus Theorem 3 says for most nonzero $x \in l_\infty$, there is a basic sequence $(e_n) \in l_\infty$ spanning a quasi-reflexive subspace of order one so that $[e_n]^{**} = \{(e_n) \cup \{\sum e_n\}\}$ and $\sum_1^k e_n$ converges $\sigma(l_\infty, l_1)$ to x. This again seems to say quasi-reflexive spaces are more common than one might expect.

2. The proof of Theorem 3 breaks down if we replace the hypothesis "X^* is separable" by "l_1 is not a subspace of X." Indeed Lemma 3 of [**1**] requires separability in a strong way. However we can obtain for each $x^{**} \in X^{**} \setminus X$ a basic sequence (e_n) in X with coefficient functionals (f_n) so that

(1) $(\sum_1^k e_n)_k$ converges $\sigma(X^{**}, X^*)$ to x^{**}.

(2) $(e_n)^{\mathrm{LIM}} = [(e_n) \cup \{\sum e_n\}]$.

(3) If $\sum b_n f_n \in (f_n)^{\mathrm{LIM}}$, then $(\sum_1^k b_n)_k$ converges.

However $(f_n)^{\mathrm{LIM}}$ could be too big to get a handle on.

3. If X has PCP but is not Polish, then X^* is nonseparable. Either l_1 is not a subspace of X and X has a 1-boundedly complete basic sequence by the remark above, or l_1 is a subspace of X and l_1 has a 1-boundedly complete basis (given by $\{u_n - u_{n+1}\}$). In either case X has a 1-boundedly complete basic sequence.

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