PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 4, December 1987

STRONG LIMIT THEOREMS FOR BLOCKWISE m-DEPENDENT AND BLOCKWISE QUASIORTHOGONAL SEQUENCES OF RANDOM VARIABLES

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(Communicated by Bert E. Fristedt)

ABSTRACT. Let $\{X_k\colon k = 1, 2, \dots\}$ be a sequence of random variables with zero mean and finite variance σ_k^2. We say that $\{X_k\}$ is blockwise m-dependent if for each p large enough the following is true: if we remove m or more consecutive X's from the dyadic block $\{X_{2^{p-1}+1}, \dots, X_{2^p}\}$, then the two remaining portions are independent. We say that $\{X_k\}$ is blockwise quasiorthogonal if for each p, the expectations $E(X_k X_l)$ are small in a certain sense again within the dyadic block $\{X_{2^{p-1}+1}, \dots, X_{2^p}\}$. Blockwise independence and blockwise orthogonality are particular cases of the above notions, respectively.

We study the a.s. behavior of the series $\sum_{k=1}^{\infty} X_k$ and that of the first arithmetic means $(1/n)\sum_{k=1}^{n} X_k$. It turns out that the classical strong limit theorems, with one exception, remain valid in this more general setting, too.

1. Introduction. Let $\{X_k\colon k = 1, 2, \dots\}$ be a sequence of random variables (in abbreviation: r.v.'s) with

$$E(X_k) = 0 \quad \text{and} \quad E(X_k^2) = \sigma_k^2 < \infty \qquad (k = 1, 2, \dots). \tag{1.1}$$

It is well known that if $\{X_k\}$ is *independent*, then the condition

$$\sum_{k=1}^{\infty} \sigma_k^2 < \infty \tag{1.2}$$

implies that the series

$$\sum_{k=1}^{\infty} X_k \quad \text{converges a.s.} \tag{1.3}$$

Hence, via the Kronecker lemma, we can conclude a strong law of large numbers. Namely, if

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty, \tag{1.4}$$

Received by the editors August 4, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60F15; Secondary 60G50.

Key words and phrases. m-dependent random variables, orthogonal random variables, first arithmetic means, strong limit theorems, strong laws of large numbers.

This reasearch was conducted while the author was a visiting professor at the University of Wisconsin, Madison, during the academic year 1985/86, on leave from the University of Szeged, Hungary.

then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to 0 \quad \text{a.s.} \quad (n \to \infty). \tag{1.5}$$

(See, e.g., [**5**, pp. 35 and 42].)

Following Hoeffding and Robbins [**2**], we say that the sequence $\{X_k\}$ of r.v.'s is *m-dependent* if $\{X_1, \dots, X_k\}$ is always independent of $\{X_l, X_{l+1}, \dots\}$ provided $l - k > m$. Here m is a fixed nonnegative integer. In particular, 0-dependence is equivalent to independence. Now it is not hard to get that implications (1.2) $\Rightarrow$ (1.3) and (1.4) $\Rightarrow$ (1.5) remain valid in the m-dependent case, as well.

On the other hand, if $\{X_k\}$ is only *orthogonal*:

$$E(X_k X_l) = 0 \qquad (k \neq l;\ k, l = 1, 2, \dots) \tag{1.6}$$

(or equivalently, *uncorrelated*), then according to the Rademacher-Menshov theorem only the stronger condition

$$\sum_{k=1}^{\infty} \sigma_k^2 [\log(k+1)]^2 < \infty \tag{1.7}$$

implies (1.3). (See, e.g., [**5**, p. 86].) In this paper the logarithms are to the base 2.

Now, applying the Kronecker lemma again yields the following. If

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} [\log(k+1)]^2 < \infty, \tag{1.8}$$

then (1.5) holds. (See, e.g., [**6**].)

It is also known that condition (1.6) can be weakened as follows. We say that $\{X_k\}$ is *quasiorthogonal* if there exists a nonrandom sequence $\{f(j) \colon j = 0, 1, \dots\}$ such that

$$|E(X_k X_l)| \leq \sigma_k \sigma_l f(|k - l|) \qquad (k, l = 1, 2, \dots) \tag{1.9}$$

and

$$\sum_{j=0}^{\infty} f(j) < \infty. \tag{1.10}$$

We note that (1.9) is equivalent to the condition

$$|\operatorname{Corr}(X_k, X_l)| \leq f(|k - l|).$$

Thus we may always assume that $f(0) = 1$ and $0 \leq f(j) \leq 1$. Now the point is that implications (1.7) $\Rightarrow$ (1.3) and (1.8) $\Rightarrow$ (1.5) remain true in the quasiorthogonal case, too. (See, e.g., [**4**].)

2. Results. The aim of this note is to provide a further weakening of independence and orthogonality, respectively. Accordingly, we introduce two new definitions.

We say that a sequence $\{X_k\}$ of r.v.'s is *blockwise m-dependent* if for each p large enough, the two sets

$$\{X_{2^{p-1}+1}, \dots, X_k\} \quad \text{and} \quad \{X_l, X_{l+1}, \dots, X_{2^p}\}$$

of r.v.'s are independent provided

$$2^{p-1} < k < l \le 2^p \quad \text{and} \quad l - k > m. \tag{2.1}$$

In particular, blockwise 0-dependence is equivalent to the requirement that the dyadic block $\{X_k: 2^{p-1} < k \le 2^p\}$ is independent for each $p = 1, 2, \ldots$. So, in this case $\{X_k\}$ can be equally called *blockwise independent.*

We remark that blockwise m-dependence implies, among others, that if $2^{p-1} < k_1 < k_2 < \cdots < k_t \le 2^p$ with some integers p and t, then the r.v.'s $X_{k_1}, X_{k_2}, \ldots, X_{k_t}$ are independent provided the gap between any two consecutive k's is larger than m, that is,

$$k_{\tau+1} - k_\tau > m \qquad (\tau = 1, 2, \ldots, t-1;\ t \ge 2).$$

We say that a sequence $\{X_k\}$ of r.v.'s is *blockwise quasiorthogonal* if for each $p \ge 1$ there exists a nonrandom sequence $\{f_p(j): j = 0, 1, \ldots, 2^{p-1} - 1\}$ such that

$$|E(X_k X_l)| \le \sigma_k \sigma_l f_p(|k - l|) \qquad (2^{p-1} < k, l \le 2^p) \tag{2.2}$$

and

$$\sum_{j=0}^{2^{p-1}-1} f_p(j) \le C \tag{2.3}$$

(cf. (1.9) and (1.10)), where C and $C_1, C_2, \ldots$ later on denote positive absolute constants.

In the special case when $f_p(j) = 1$ for $j = 0$ and $f_p(j) = 0$ for $j = 1, 2, \ldots$, for each $p = 1, 2, \ldots$, we call $\{X_k\}$ *blockwise orthogonal.* That is, in this case

$$E(X_k X_l) = 0 \qquad (k \ne l;\ 2^{p-1} < k, l \le 2^p;\ p \ge 1).$$

We will prove two theorems.

THEOREM 1. *If $\{X_k\}$ is blockwise m-dependent, then condition* (1.4) *implies* (1.5).

THEOREM 2. *If $\{X_k\}$ is blockwise quasiorthogonal, then condition* (1.7) *implies* (1.3).

Now the Kronecker lemma and Theorem 2 yield the following.

COROLLARY 1. *If $\{X_k\}$ is blockwise quasiorthogonal, then condition* (1.8) *implies* (1.5).

REMARK 1. It is interesting to observe that implication (1.2) $\Rightarrow$ (1.3) fails to hold in the blockwise m-dependent case. A counterexample for $m = 0$ is the following. Let $\{X_k\}$ be a sequence of independent r.v.'s such that conditions (1.1) and (1.2) are satisfied and, in addition, $X_1 \ne 0$ a.s. We define a new sequence $\{Y_k\}$ by

$$Y_k = \begin{cases} \frac{1}{p} X_1 & \text{if } k = 2^p + 1 \ (p = 1, 2, \ldots), \\ X_k & \text{otherwise.} \end{cases}$$

Then $\{Y_k\}$ is clearly blockwise independent, $EY_k = 0$,

$$\sum_{k=1}^{\infty} E(Y_k^2) = \sigma_1^2 \sum_{p=1}^{\infty} \frac{1}{p^2} + \sum_{\substack{k=1 \\ k \ne 2^p+1}}^{\infty} \sigma_k^2 < \infty,$$

and the partial sums

$$\sum_{k=1}^{n} X_k = X_1 \sum_{l=1}^{q-1} \frac{1}{l} + \sum_{\substack{k=1 \\ k\neq 2^p+1}}^{n} X_k \qquad (2^{q-1} < n \le 2^q)$$

diverge a.s.

REMARK 2. It will turn out from the proofs in §3 that if in the definitions of "blockwise m-dependence" and "blockwise quasiorthogonality" we replace the dyadic blocks $2^{p-1} < k \le 2^p$ by other blocks $\theta^{p-1} < k \le \theta^p$, where $\theta > 1$ is any fixed number, we do have results analogous to those expressed in Theorems 1 and 2 and Corollary 1.

On the other hand, we conjecture that if in the definitions we substitute the block $p^\alpha < k \le (p+1)^\alpha$ for $2^{p-1} < k \le 2^p$, where $\alpha > 1$ is fixed, then Theorems 1 and 2 and Corollary 1 are no longer true. For brevity, here we state two of our guesses.

CONJECTURE 1. *For every $\alpha > 1$, there exists a sequence $\{X_k\}$ of r.v.'s such that conditions* (1.1) *and* (1.4) *are satisfied, $\{X_k\colon p^\alpha < k \le (p+1)^\alpha\}$ is independent for each $p = 1, 2, \dots,$ but*

$$\limsup_{n\to\infty} \frac{1}{n} \left| \sum_{k=1}^{n} X_k \right| = \infty \quad a.s. \tag{2.4}$$

CONJECTURE 2. *For every $\alpha > 1$, there exists a sequence $\{X_k\}$ of r.v.'s such that conditions* (1.1) *and* (1.8) *are satisfied,*

$$E(X_k X_l) = 0 \qquad (k \ne l;\ p^\alpha < k, l \le (p+1)^\alpha;\ p \ge 1),$$

and (2.4) *holds.*

3. Proofs. They follow standard techniques combining with appropriate modifications.

PROOF OF THEOREM 1. First we show that

$$E\left[\sum_{k=2^{p-1}+1}^{2^p} X_k\right]^2 \le C_1 \sum_{k=2^{p-1}+1}^{2^p} \sigma_k^2. \tag{3.1}$$

To achieve this moment inequality, we only need orthogonality of X_k and X_l with $|k-l| > m$ and $2^{p-1} < k, l < 2^p$. Indeed, using the Cauchy inequality both for integrals and for numerical sequences, we get

$$\begin{aligned} E\left[\sum_{k=2^{p-1}+1}^{2^p} X_k\right]^2 &= \sum_{k=2^{p-1}+1}^{2^p} E(X_k^2) + 2\sum_{j=1}^{m} \sum_{k=2^{p-1}+1}^{2^p-j} E(X_k X_{k+j}) \\ &\le \sum_{k=2^{p-1}+1}^{2^p} \sigma_k^2 + 2\sum_{j=1}^{m} \sum_{k=2^{p-1}+1}^{2^p-j} \sigma_k \sigma_{k+j} \\ &\le (1+2m) \sum_{k=2^{p-1}+1}^{2^p} \sigma_k^2, \end{aligned} \tag{3.2}$$

proving (3.1). Hence, by (1.4),

$$\sum_{p=1}^{\infty} E\left[\frac{1}{2^p}\sum_{k=2^{p-1}+1}^{2^p} X_k\right]^2 \le C_1 \sum_{p=1}^{\infty} \frac{1}{2^{2p}} \sum_{k=2^{p-1}+1}^{2^p} \sigma_k^2 \le C_1 \sum_{k=2}^{\infty} \frac{\sigma_k^2}{k^2} < \infty$$

and the dominated convergence theorem implies

$$\frac{1}{2^p}\sum_{k=2^{p-1}+1}^{2^p} X_k \to 0 \quad \text{a.s.} \quad (p \to \infty). \tag{3.3}$$

The representation

$$\frac{1}{2^p}\sum_{k=1}^{2^p} X_k = \frac{1}{2^p} X_1 + \sum_{q=1}^{p} \frac{1}{2^{p-q}} \left(\frac{1}{2^q} \sum_{k=2^{q-1}+1}^{2^q} X_k\right) \tag{3.4}$$

reveals that the first arithmetic mean in question is a linear mean of the quantities occurring in the limit relation (3.3). In fact, the right-hand side in (3.4) is a linear mean induced by a positive, permanent summation process (see, e.g., [**1**, p. 65]), and consequently, the limit of (3.4) also exists and is the same as that of (3.3):

$$\frac{1}{2^p}\sum_{k=1}^{2^p} X_k \to 0 \quad \text{a.s.} \quad (p \to \infty). \tag{3.5}$$

Second, we prove that

$$\max_{2^{p-1}<n\le 2^p} \frac{1}{2^p}\left|\sum_{k=2^{p-1}+1}^{n} X_k\right| \to 0 \quad \text{a.s.} \quad (p \to \infty). \tag{3.6}$$

To this effect, we split the sum in (3.6) into $m+1$ subsums as follows

$$\sum_{k=2^{p-1}+1}^{n} X_k = \sum_{j=1}^{m+1} \sum_{l=0}^{l_j(n)} X_{2^{p-1}+l(m+1)+j}, \tag{3.7}$$

where $l_j(n)$ is defined by the condition

$$2^{p-1} + (m+1)l_j(n) + j \le n < 2^{p-1} + (m+1)(l_j(n)+1) + j \qquad (j = 1, 2, \ldots, m+1).$$

This certainly makes sense if p is large enough, say $p \ge p_0$.

According to the remark made after the definition of blockwise m-dependence in §2, each of the inner sums on the right-hand side of (3.7) consists of independent r.v.'s. Thus, we can apply the well-know Kolmogorov inequality $(m+1)$ times separately. As a result, we obtain for every $\varepsilon > 0$ that

$$P\left[\max_{2^{p-1}<n\le 2^p} \frac{1}{2^p}\left|\sum_{k=2^{p-1}+1}^{n} X_k\right| > \varepsilon\right]$$
$$\le \sum_{j=1}^{m+1} P\left[\max_{0\le L\le l_j(2^p)} \left|\sum_{l=0}^{L} X_{2^{p-1}+l(m+1)+j}\right| \ge \frac{2^p\varepsilon}{m+1}\right]$$
$$\le \frac{(m+1)^2}{2^{2p}\varepsilon^2} \sum_{j=1}^{m+1} \sum_{l=0}^{l_j(2^p)} \sigma^2_{2^{p-1}+l(m+1)+j} = \frac{(m+1)^2}{2^{2p}\varepsilon^2} \sum_{k=2^{p-1}+1}^{2^p} \sigma_k^2,$$

whence, by (1.4),

$$\sum_{p=p_0}^{\infty} P\left[\max_{2^{p-1}<n\le 2^p}\left|\sum_{k=2^{p-1}+1}^{2^p} X_k\right| > \varepsilon\right] \le \frac{4(m+1)^2}{\varepsilon^2}\sum_{k=2^{p_0-1}+1}^{\infty}\frac{\sigma_k^2}{k^2} < \infty.$$

Thus, the Borel-Cantelli lemma implies (3.6). Combining (3.5) and (3.6) yields (1.5).

PROOF OF THEOREM 2. We begin with the moment inequality

$$E\left[\sum_{k=a+1}^{a+n} X_k\right]^2 \le C_2 \sum_{k=a+1}^{a+n}\sigma_k^2, \tag{3.8}$$

where the positive integers a and n are such that they belong to the same dyadic block: $2^{p-1} < a+1 \le a+n \le 2^p$ with some $p \ge 1$. In fact, we can proceed similarly to (3.2): by (1.9),

$$\begin{aligned} E\left[\sum_{k=a+1}^{a+n} X_k\right]^2 &= \sum_{k=a+1}^{a+n} E(X_k^2) + 2\sum_{j=1}^{n-1}\sum_{k=a+1}^{a+n-j} E(X_k X_{k+j}) \\ &\le \sum_{k=a+1}^{a+n}\sigma_k^2 + 2\sum_{j=1}^{n-1}\sum_{k=a+1}^{a+n-j}\sigma_k\sigma_{k+j} f_p(j) \\ &\le \left(1 + 2\sum_{j=1}^{n-1} f_p(j)\right)\sum_{k=a+1}^{a+n}\sigma_k^2, \end{aligned}$$

which is (3.8), using (1.10).

Now we are able to show that

$$\sum_{k=1}^{2^p} X_k = X_1 + \sum_{q=1}^{p-1}\sum_{k=2^{q-1}+1}^{2^q} X_k \quad \text{converges a.s.} \quad (p\to\infty). \tag{3.9}$$

This follows from the dominated convergence theorem, since by the Cauchy inequality, (3.8), and (1.7), we have

$$\begin{aligned} \left[\sum_{q=1}^{\infty} E\left|\sum_{k=2^{q-1}+1}^{2^q} X_k\right|\right]^2 &\le \sum_{q=1}^{\infty} q^2 E\left[\sum_{k=2^{q-1}+1}^{2^q} X_k\right]^2 \sum_{q=1}^{\infty}\frac{1}{q^2} \\ &\le \frac{C_2\pi^2}{6}\sum_{q=1}^{\infty} q^2 \sum_{k=2^{q-1}+1}^{2^q}\sigma_k^2 \\ &\le \frac{2C_2\pi^2}{3}\sum_{k=2}^{\infty}\sigma_k^2[\log k]^2 < \infty. \end{aligned}$$

Finally, we have to estimate the fluctuation within a block $2^{p-1} < n \le 2^p$. From the moment inequality (3.8) we can deduce a Rademacher-Menshov type maximal inequality (see, e.g., [**3**]):

$$E\left[\max_{2^{p-1}<n\le 2^p}\left|\sum_{k=2^{p-1}+1}^{n} X_k\right|\right]^2 \le C_2[\log 2^p]^2 \sum_{k=2^{p-1}+1}^{2^p}\sigma_k^2.$$

Hence again the dominated convergence theorem implies, via (1.7), that

$$\max_{2^{p-1}<n\leq 2^p}\left|\sum_{k=2^{p-1}+1}^{n} X_k\right| \to 0 \quad \text{a.s.} \quad (p\to\infty). \tag{3.10}$$

Putting (3.9) and (3.10) together results in (1.3).

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