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ZERO SPAN IS A SEQUENTIAL STRONG WHITNEY-REVERSIBLE PROPERTY

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Dedicated to Professor Jun-ichi Nagata on his 60th birthday

ABSTRACT. The concept of span of metric spaces was introduced by Lelek [**5**]. Span is an important concept in regard to chainability of continua. In this paper, motivated by recent results [**2, 11**], we show that zero span is a sequential strong Whitney-reversible property.

1. Introduction. All spaces are assumed to be nonempty *metric* spaces, and all maps to be continuous functions. A *continuum* is a compact connected space and a *Peano continuum* is a locally connected continuum. The *hyperspace* of a continuum X is the space

$$C(X) = \{A \subseteq X | A \text{ is nonempty, compact and connected}\}$$

metrized with the Hausordff metric [**9**]. Let $F_1(X) = \{\{x\} | x \in X\}$. A *Whitney map* for $C(X)$ is a map $\mu: C(X) \to [0, \infty)$ such that

(i) $\mu(\{x\}) = 0$ for each $x \in x$, and

(ii) if $A, B \in C(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$.

Namely, a Whitney map is a size function, in some sense. For a Whitney map μ for $C(X)$, Whitney levels $\mu^{-1}(t)$, $0 \leq t \leq \mu(X)$ are coverings of X which, as t gets to close to zero, converge to $\mu^{-1}(0) = F_1(X) \cong X$. Therefore, it is interesting to obtain information about the structure of Whitney levels, and to determine those properties which are preserved by the convergence of positive Whitney levels $\mu^{-1}(t)$, $0 < t \leq \mu(X)$, to the zero level. A topological property $\mathcal{P}$ is said to be *sequentially strongly Whitney-reversible* (resp., *strongly Whitney-reversible*) [**9, 10**], provided that if a continuum X has a Whitney map μ for $C(X)$ such that there exists a decreasing sequence $\{t_n\}_{n\geq 1}$ in $(0, \mu(X)]$ such that $\lim_n t_n = 0$ and each $\mu^{-1}(t_n)$, $n \geq 1$, has property $\mathcal{P}$ (resp., $\mu^{-1}(t)$ has property $\mathcal{P}$ for every $0 < t < \mu(X)$), then X has property $\mathcal{P}$. Clearly, every sequential strong Whitney-reversible property is a strong Whitney-reversible property. In [**2, 4, 9 and 10**], a number of properties were shown to be (sequentially) strongly Whitney-reversible, e.g., chainability, tree-likeness, acyclicity, dim $\leq n$, being atriodic, etc. In fact, it is

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interesting and important to determine whether or not a given topological property is (sequentially) strongly Whitney-reversible.

The purpose of this paper is to show that zero span is a sequential strong Whitney-reversible property.

2. Definitions and preliminaries. Let X be a connected space with a metric $\mathbf{d}$, and let $\pi_i: X \times X \to X$ denote the ith coordinate projection for $i = 1, 2$. The *surjective span* $\sigma^*(X)$ (resp., *surjective semispan* $\sigma_0^*(X)$) of X is the least upper bound of the set of all real numbers α which satisfy the following condition:

There exists a connected subset $C \subseteq X \times X$ such that $\pi_1(C) = \pi_2(C) = X$ (resp., $\pi_1(C) = X$) and $\mathbf{d}(x, y) \geq \alpha$ for each $(x, y) \in C$.

The *span* $\sigma(X)$ and the *semispan* $\sigma_0(X)$ of X, not necessarily connected, are

$$\sigma(X) = \sup\{\sigma^*(A) | A \subseteq X,\ A \neq \emptyset \text{ and connected}\}, \text{ and}$$
$$\sigma_0(X) = \sup\{\sigma_0^*(A) | A \subseteq X,\ A \neq \emptyset \text{ and connected}\}.$$

It is known that every chainable continuum has zero span (see [**5**]). Recently, Oversteegen and Tymchatyn [**11**] characterized continua X with $\sigma^*(X) = 0$ in terms of uniformizations of two sequences of arcs converging in $C(X)$ to X. In this paper, we will essentially use their characterization. Let I be the *closed unit interval* $[0, 1]$. For maps $f, g: I \to X$ and a positive number $\varepsilon > 0$, we say that g can be *ε-uniformized with a piece of f* if there exist maps $a, b: I \to I$ such that $b(I) = I$ and $\mathbf{d}(f \circ a(t), g \circ b(t)) < \varepsilon$ for each $t \in I$.

LEMMA 1 [**11**, THEOREM 4]. (1) *Let $X \subseteq Y$ be continua and let $\varepsilon > 0$ be a positive number. If $\sigma^*(X) < \varepsilon$, then the following condition $(*)_\varepsilon$ is satisfied:*

$(*)_\varepsilon$ *For each pair of sequences $\{f_i\}_{i\geq 1}$, $\{g_i\}_{i\geq 1}$ of maps $f_i, g_i: I \to Y$ such that $\operatorname{Lim}_i f_i(I) = \operatorname{Lim}_i g_i(I) = X$, there exists an integer $n \geq 1$ such that for each $i \geq n$, either f_i is ε-uniformized with a piece of g_i or g_i is ε-uniformized with a piece of f_i.*

(2) *Let X be a continuum in a Peano continuum Y. If for each positive number $\varepsilon > 0$, the condition $(*)_\varepsilon$ is satisfied, then $\sigma^*(X) = 0$.*

Some continuity property of spans is useful. The following is obtained by a slight modification of the proof of [**7**, 3.1].

LEMMA 2. *Let $X \subseteq Y$ be continua and let $\varepsilon > 0$ be a positive number. If there exists a sequence $\{X_n\}_{n\geq 1}$ of continua in Y such that $\operatorname{Lim}_n X_n = X$ and $\tau(X_n) \geq \varepsilon$ for each $n \geq 1$, then $\tau(X) \geq \varepsilon$, where $\tau = \sigma^*, \sigma_0^*, \sigma$, or σ_0.*

Basic facts about Whitney maps and Whitney levels may be found in [**9**].

3. Main results. In this section we will show that zero span is a sequential strong Whitney-reversible property. In fact, we will obtain a more general result, as follows:

THEOREM. *Let X be a continuum and let μ be any Whitney map for $C(X)$. If there exists a decreasing sequence $\{t_n\}_{n\geq 1}$ in $(0, \mu(X)]$ such that $\lim_n t_n = t$ and $\sigma(\mu^{-1}(t_n)) = 0$ for each $n \geq 1$, then $\sigma(\mu^{-1}(t)) = 0$.*

PROOF. Since zero span is a topological invariant in the class of continua, we may assume that X is a subcontinuum of the Hilbert cube $\mathbf{Q}$ with a metric $\mathbf{d}$. By

[12], there exists a Whitney map $\tilde{\mu}$ for $C(\mathbf{Q})$ such that $\tilde{\mu}|C(X) = \mu$. Then $\tilde{\mu}^{-1}(t)$ is a Peano continuum. Hence, applying Lemma 1, we will show that, for every nonempty subcontinuum Λ of $\mu^{-1}(t)$, $\sigma^*(\Lambda) = 0$.

Let $\{f_i\}_{i\geq 1}$, $\{g_i\}_{i\geq 1}$ be a given pair of sequences of maps $f_i, g_i: I \to \tilde{\mu}^{-1}(t)$ such that $\mathrm{Lim}_i f_i(I) = \mathrm{Lim}_i g_i(I) = \Lambda$ and let $\varepsilon > 0$ be a positive number. By [3], there exists a positive number $\eta > 0$ such that

$$\text{if } A, B \in C(\mathbf{Q}),\ A \subseteq B, \text{ and } \tilde{\mu}(B) - \tilde{\mu}(A) < \eta, \text{ then } \mathbf{d}_H(A, B) < \varepsilon/3,$$

where $\mathbf{d}_H$ is the Hausdorff metric on $C(\mathbf{Q})$ induced by the metric $\mathbf{d}$. Since $\lim_n t_n = t$, there exists an integer $n \geq 1$ such that $t_n - t < \eta$. Then

$$\text{(1)} \qquad \text{if } A \in \tilde{\mu}^{-1}(t),\ B \in \tilde{\mu}^{-1}(t_n), \text{ and } A \subseteq B, \text{ then } \mathbf{d}_H(A, B) < \varepsilon/3.$$

Let take a decreasing sequence $\{X_k\}_{k\geq 1}$ of Peano continua in $\mathbf{Q}$ such that for each $k \geq 1$, X_k is a closed neighborhood of X_{k+1} in $\mathbf{Q}$ and $\bigcap_{k\geq 1} X_k = X$. Then we can easily see that

$$\text{(2)} \qquad \mathrm{int}_{\tilde{\mu}^{-1}(t_n)}(\tilde{\mu}^{-1}(t_n) \cap C(X_k)) \supseteq \tilde{\mu}^{-1}(t_n) \cap C(X_{k+1}) \quad \text{for each } k \geq 1,$$

and

$$\text{(3)} \qquad \bigcap_{k\geq 1} (\tilde{\mu}^{-1}(t_n) \cap C(X_k)) = \mu^{-1}(t_n).$$

Since $\sigma(\mu^{-1}(t_n)) = 0$, by (2), (3), and Lemma 2, there exists an integer $k \geq 1$ such that

$$\text{(4)} \qquad \sigma(\tilde{\mu}^{-1}(t_n) \cap C(X_k)) < \varepsilon/3.$$

Then by (2) and (3), passing to subsequences if necessary, we may assume that $f_i(I), g_i(I) \subseteq \tilde{\mu}^{-1}(t_n) \cap C(X_k)$ for each $i \geq 1$.

Since X_k is a Peano continuum, X_k admits a convex metric ρ. Then a function $K: C(X_k) \times [0, \infty) \to C(X_k)$ given by

$$K(A, s) = \{x \in X_k | \rho(x, a) \leq s \text{ for some } a \in A\} \quad \text{for } A \in C(X_k) \text{ and } s \in [0, \infty)$$

is continuous (see [8]). Hence we have the map $h: \tilde{\mu}^{-1}(t) \cap C(X_k) \to \tilde{\mu}^{-1}(t_n) \cap C(X_k)$ defined by, for each $A \in \tilde{\mu}^{-1}(t) \cap C(X_k)$,

$$\text{(5)} \qquad h(A) = K(A, \alpha(A, t_n)), \quad \text{where } \tilde{\mu}(K(A, \alpha(A, t_n))) = t_n$$

(see [2 and 10]).

Let consider the pair of sequences $\{f_i^*\}_{i\geq 1}$, $\{g_i^*\}_{i\geq 1}$ of maps $f_i^* = h \circ f_i$, $g_i^* = h \circ g_i: I \to \tilde{\mu}^{-1}(t_n) \cap C(X_k)$. Then $\mathrm{Lim}_i f_i^*(I) = \mathrm{Lim}_i g_i^*(I) = h(\Lambda)$. By (4), $\sigma^*(h(\Lambda)) \leq \sigma(\tilde{\mu}^{-1}(t_n) \cap C(X_k)) < \varepsilon/3$. Hence by Lemma 1(1), there exists an integer $i_0 \geq 1$ such that

> for each $i \geq i_0$, either f_i^* is $\varepsilon/3$-uniformized with a piece of g_i^* or g_i^* is $\varepsilon/3$-uniformized with a piece of f_i^*.

For an arbitrary integer $i \geq i_0$, assume that f_i^* is $\varepsilon/3$-uniformized with a piece of g_i^*. Thus, there exists maps $a_i, b_i: I \to I$ such that

$$\text{(6)} \qquad a_i(I) = I,$$

and

$$\mathbf{d}_H(f_i^* \circ a_i(s), g_i^* \circ b_i(s)) < \varepsilon/3 \quad \text{for each } s \in I. \tag{7}$$

For each $s \in I$, $f_i \circ a_i(s) \in \tilde{\mu}^{-1}(t)$, and by (5),

$$f_i \circ a_i(s) \subseteq h(f_i \circ a_i(s)) = f_i^* \circ a_i(s) \in \tilde{\mu}^{-1}(t_n).$$

Hence, by (1),

$$\mathbf{d}_H(f_i \circ a_i(s), f_i^* \circ a_i(s)) < \varepsilon/3 \quad \text{for each } s \in I. \tag{8}$$

Similarly, we can see that

$$\mathbf{d}_H(g_i \circ b_i(s), g_i^* \circ b_i(s)) < \varepsilon/3 \quad \text{for each } s \in I. \tag{9}$$

Hence, by (7), (8) and (9),

$$\begin{aligned} \mathbf{d}_H(f_i \circ a_i(s), g_i \circ b_i(s)) &\leq \mathbf{d}_H(f_i \circ a_i(s), f_i^* \circ a_i(s)) \\ &\quad + \mathbf{d}_H(f_i^* \circ a_i(s), g_i^* \circ b_i(s)) \\ &\quad + \mathbf{d}_H(g_i^* \circ b_i(s), g_i \circ b_i(s)) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon \quad \text{for each } s \in I. \end{aligned} \tag{10}$$

It follows from (6) and (10) that f_i is ε-uniformized with a piece of g_i.

Similarly, we can see that for $i \geq i_0$, if g_i^* is $\varepsilon/3$-uniformized with a piece of f_i^*, then g_i is ε-uniformized with a piece of f_i. Therefore, for each $i \geq i_0$, either f_i is ε-uniformized with a piece of g_i or g_i is ε-uniformized with a piece of f_i. From Lemma 1(2), we have that $\sigma^*(\Lambda) = 0$. It follows that $\sigma(\mu^{-1}(t)) = \sup\{\sigma^* \in \Lambda) | \Lambda \in C(\mu^{-1}(t))\} = 0$.

COROLLARY 1. *Zero span is a sequential strong Whitney-reversible property.*

By [**11**], continua with zero surjective semispan are weakly chainable, atriodic and tree-like. Hence we have

COROLLARY 2. *Let X be a continuum. If there exist a Whitney map μ for $C(X)$ and a decreasing sequence $\{t_n\}_{n\geq 1}$ in $(0, \mu(X)]$ such that $\lim_n t_n = 0$ and $\sigma(\mu^{-1}(t_n)) = 0$ for each $n \geq 1$, then X is weakly chainable, atriodic, and tree-like.*

By [**1**], we know that for a continuum X, $\sigma(X) = 0$ if and only if $\sigma_0(X) = 0$. Hence zero semispan is also a sequential strong Whitney-reversible property. However, in the proof of the Theorem, the map h need not be surjective. Therefore we do not know *whether or not zero surjective (semi)span is a sequential strong Whitney-reversible property.*

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