PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 4, December 1987

A FINITENESS PROBLEM FOR ONE DIMENSIONAL MAPS

W. DE MELO

(Communicated by Haynes R. Miller)

ABSTRACT. We discuss the connection between the density of structurally stable maps in the space of unimodal maps of the interval, the finiteness of attractors and the nonexistence of wandering intervals. We show that in the space of unimodal maps having an eventually periodic flat critical point, there is a residual subset whose maps have infinitely many sinks. In this space there are also maps having a wandering interval.

In [**Su**] Sullivan solved a finiteness problem for endomorphisms of the Riemann sphere and this was one of the main ingredients in [**MSS**] for the proof of the density of structural stability in the space of rational maps. Here we formulate a similar finiteness problem for C^∞ endomorphisms of the circle and of the compact interval $[0,1]$ and we discuss the connection of this problem with the density of structural stability.

Before stating our results let us recall Sullivan's theorem in order to stress the similarity of the two situations. If $f\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is an endomorphism of the Riemann sphere $\overline{\mathbf{C}}$, we have a decomposition $\overline{\mathbf{C}} = J(f) \cup F(f)$ where $J(f) = \overline{\mathbf{C}} - F(f)$ and $F(f)$ is the domain of normality, i.e., a point belongs to $F(f)$ if it has a neighborhood where the family of iterates of f is a normal family. The open set $F = F(f)$ is the Fatou set of f and $J(f)$ is the Julia set. Both are f-invariant sets: $f(F) = f^{-1}(F) = F$. Hence the image of a connected component of F is another connected component. As a consequence, the forward orbit of a component of F is either infinite or some iterate of it is a periodic component. In the last case we say that the original component is eventually periodic. Sullivan's theorem states that all components are eventually periodic and that there are only finitely many periodic components.

Let M denote either the circle S^1 or the compact interval $[0,1]$ and $\operatorname{End}(M)$ be the set of C^∞ maps of M into M endowed with the C^∞ topology. Let $C(f) = \{x \in M; Df(x) = 0\} \cup \partial M$. We will consider only maps f such that $C(f)$ is finite. This clearly contains an open and dense subset of $\operatorname{End}(M)$.

DEFINITION. For $f \in \operatorname{End}(M)$ with $C(f)$ finite, let $J(f) = \{x \in M; \exists$ sequences $y_i \to x$ and $n_i \to \infty$ s.t. $f^{n_i}(y_i) \in C(f)\}$, $F(f) = M - J(f)$.

Clearly $J(f)$ is a closed, forward invariant set, i.e., $f(J) \subset J$. It is not, in general, backward invariant. However, if $x \in M$ is such that $f^n(x) \in J(f)$ and $f^i(x) \notin C(f)$ for $0 \le i < n$ then $x \in J(f)$. Similarly, $F(f)$ is an open, backward

Received by the editors June 16, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F10; Secondary 58F13.

Key words and phrases. Unimodal maps, structural stability, wandering intervals, sinks, attractors, critical point.

Partially supported by CNPq-NSF agreement.

invariant set which is not, in general, forward invariant. If $D \subset F$ is a connected component of F then $f(D)$ is contained in the closure of a connected component and it is in fact a component if $D \cap C(f) = \varnothing$. Let D_0 be a component of F and D_i be the component whose closure contains $f^i(D_0)$. If there is an $n \in \mathbf{N}$ such that $D_n = D_0$ we say that D_0 is a periodic domain of f. If $D_i \cap D_j = \varnothing$ for $i \neq j$ we say that D_0 is a wandering domain. It is clear that if D_0 is not a wandering domain then D_{n_0} is a periodic domain for some $n_0 \in \mathbf{N}$. In this case we say that D_0 is an eventually periodic domain.

The dynamics of f in the periodic domains are very simple and are described below.

PROPOSITION 1. *If D is a periodic domain of f of period p, then f^p maps $\overline{D}$ into $\overline{D}$, every point of $\overline{D}$ is asymptotic to a periodic point and the periods of the periodic points of f in D are bounded.*

In particular, if all the periodic points of f are hyperbolic (i.e. the graphs of all iterates of f are transversal to the diagonal) then f has a finite number of periodic points in D.

Let $\mathcal{F}_0 = \mathcal{F}_0(M)$ be the set of $f \in \mathrm{End}(M)$ having only finitely many periodic domains, and $\mathcal{F}_1$ the set of maps having no wandering domains. The question is: how big is the set $\mathcal{F}_0 \cap \mathcal{F}_1$? In Proposition 3 we show that $\mathcal{F}_0 \cap \mathcal{F}_1$ does not contain all the maps f with $C(f)$ finite. Fatou proved in [**F**] that if f is a polynomial then f has finitely many attractors. Hence $f \in \mathcal{F}_0$ and $\mathcal{F}_0$ is dense in $\mathrm{End}(M)$. We do not know if $\mathcal{F}_1$ is dense in $\mathrm{End}(M)$. If f has negative Schwarzian derivative (i.e., $Sf(x) = f'''(x)/f'(x) - \frac{3}{2}(f''(x)/f'(x))^2 < 0$ for all x) then Singer proved in [**S**] that f has finitely many attractors and hence $f \in \mathcal{F}_0$. If $M = [0,1]$, $C(f)$ has only one point in $]0,1[$ and $S(f) < 0$ then $f \in \mathcal{F}_0 \cap \mathcal{F}_1$, [**Mi**, **G**]. If $J(f) \cap C(f) = \varnothing$ it follows from [**Ma**] that $f \in \mathcal{F}_0 \cap \mathcal{F}_1$. Finally, if $f\colon [0,1] \to [0,1]$ has only one critical point which is nonflat then $f \in \mathcal{F}_1$ [**dM-Str**]. For maps with more than one critical point this is still an open question even if $S(f) < 0$.

The following proposition and its corollary connects the finiteness problem with the density of structural stability.

We say that f is semiconjugate to g if there is a continuous, monotone map h of M onto M such that $hf = gh$. If h is a homeomorphism we say that f is conjugate to g. A map f is structurally stable if it is conjugate to all nearby maps. We denote by $\Sigma(M)$ the set of structurally stable maps.

PROPOSITION 2. *Let $\mathcal{U}$ be the interior of $\mathcal{F}_0 \cap \mathcal{F}_1$. Let $\mathcal{V} \subset \mathrm{End}(M)$ be the interior of the set of maps f such that if $x, y \in C(f)$ then the orbit of x is disjoint from the orbit of y. Then $\Sigma(M)$ is open and dense in $\mathcal{U} \cap \mathcal{V}$.*

COROLLARY 1. *Let $\mathcal{S}$ be the set of maps $f\colon [0,1] \to [0,1]$ such that $f(0) = f(1) = 0$ and f has a unique critical point. Then $\Sigma([0,1])$ is dense in $\mathcal{U}([0,1])$.*

COROLLARY 2. *If $\mathcal{S}_0 \subset \mathcal{S}$ is the open set of maps having negative Schwarzian derivative then $\Sigma(M)$ is dense in $\mathcal{S}_0$.*

REMARK. The above results are similar to those of [**MSS**]. Here, the order structure of the real line plays the same role as complex differentiability in [**MSS**].

PROPOSITION 3. *Let $f \in \mathcal{U}(M)$ be such that $J(f)$ contains a unique critical point $c(f)$ which is nonwandering and eventually periodic. Then:*

(a) *There is a map $g \in \mathrm{End}(M)$ such that $C(g)$ is finite, g is semiconjugate to f and g has infinitely many periodic domains.*

(b) *There is a map $g \in \mathrm{End}(M)$ such that $C(g)$ is finite, g is semiconjugate to f and g has a wandering domain.*

REMARK. The map g in Proposition 3 is C^∞ flat at the critical point. In [**H**] Hall constructed an example of a C^∞ homemorphism of the circle which has a wandering domain. See also [**S-I**] for a different construction of maps with infinitely many sinks and wandering domains.

PROBLEM. If $f \in \mathrm{End}(M)$ is such that no critical point of f is flat then $f \in \mathcal{F}_0 \cap \mathcal{F}_1$.

REMARK. A well-known conjecture is the density of the set $\mathcal{A}$ of maps whose critical set is in the basin of the attractors. Clearly $\mathcal{A} \subset \mathcal{V}$. The density of $\mathcal{A}$ would imply by [**Ma**] the density of $\Sigma(M)$. For endomorphisms of the Riemann sphere, the density of $\mathcal{A}$ is still a difficult open question. Corollary 2 above shows that in order to prove the density of $\Sigma(M)$ for unimodal maps we can avoid this problem as in [**M-S-S**]. However this question cannot be avoided for maps with several critical points as the example in §2 indicates.

We are thankful to C. Gutiererez and D. Sullivan for helpful conversations.

2. Proof of the propositions. Proposition 1 follows easily from some lemmas in [**Y**]. In fact, let D be a periodic domain of period k for f. Then $g = f^k$ maps $\overline{D}$ into $\overline{D}$ and all limit points of the backward orbits of the critical points of g in D are contained in the boundary of D. Hence, by Lemma 1 of [**Y**], each critical point c_i is either periodic or wandering. Thus all ponts in the backward orbit of the critical points are wandering. By Lemma 4 of [**Y**] every nonwandering point x is periodic. Therefore every point in D is asymptotic to a periodic point. If the periods of periodic points in D are not bounded there is a sequence x_i of periodic points of g of period n_i such that $n_i \to \infty$ and x_i converges to $x \in D$. Clearly x is nonwandering and if $]a,b[$ is the maximal interval containing x which is disjoint from the backward orbits of the critical points we have that $g^N(]a,b[) =]a,b[$ for some integer $N > 0$. Since g^N is monotone in $]a,b[$, all periodic points of g in $]a,b[$ are fixed points of g^{2N}. This is a contradiction and proves Proposition 1.

REMARK. If all periodic points of f in a periodic domain D are hyperbolic then the number of periodic points in D is finite. If all periodic points of f are hyperbolic and the number of attractors is finite then the number of periodic domains is also finite.

To prove Proposition 2 we will need some lemmas.

LEMMA 1. *Let S_1 and S_2 be subsets of M with S_2 dense in M. If $h_0: S_1 \to S_2$ is a monotone map then there is a unique continuous, monotone and surjective map $h: M \to M$ such that $h|S_1 = h_0$.*

PROOF. Left to the reader.

LEMMA 2. *Let f_t, $t \in [0,1]$, be a continuous one-parameter family of maps such that*

(a) *$C(f_t)$ is finite and varies continuously with t;*

(b) $x, y \in C(f_t) \Rightarrow \theta_+(x, f_t) \cap \theta_+(y, f_t) = \varnothing$ where $\theta_+(x, f_t) = \bigcup_{n \geq 0} f_t^n(x)$ is the forward orbit of x.

Then, for each $t \in [0,1]$, there is a monotone bijection

$$h_t\colon \theta_-(C(f_0)) \to \theta_-(C(f_t))$$

such that $h_t f_0 = f_t h_t$. Here $\theta_-(C(f_t)) = \bigcup_{n \geq 0} f_t^{-n}(C(f_t))$ is the backward orbit of the set $C(f_t)$.

PROOF. Let $x_0 \in M$ and $n \in \mathbf{N}$ be such that $f^n(x_0) = c_0 \in C(f_0)$. Let $t \mapsto c(t)$ be a continuous curve such that $c_0 = c(0)$ and $c(t) \in C(f_t)$ for all $t \in [0,1]$. Let $I(x_0) = \{s \in [0,1]$; there is a continuous curve $x\colon [0,s] \to M$ such that $f^n(x(t)) = c(t)$ and $x(0) = x_0\}$. Since $\theta_-(c(t)) \cap C(f_t) = \varnothing$ for every $t \in [0,1]$, it follows from the implicit function theorem that $I(x_0) = [0,1]$. Set $h_t(x_0) = x_t$ and $h_t(c_0) = c_t$. It is easy to check that h_t satisfies the conditions of Lemma 2.

LEMMA 3. *Let f_t, $t \in [0,1]$, be a continuous family of maps in* End(M) *satisfying the following properties:*

(1) The critical points of f_t are nondegenerate and they have disjoint orbits and no critical point is eventually periodic;

(2) $D_{i,j}^t$, $i = 1, \ldots, N$; $j = 1, \ldots, p_i$, are periodic domains of f_t whose endpoints depend continuously on t;

(3) the periodic points of f_t in $\overline{D_{i,j}^t}$ are hyperbolic.

Then there is a monotone bijection

$$h_t\colon C(f_0) \cup_{n,i,j} f_0^{-n}(D_{i,j}^0) \to C(f_t) \cup_{n,i,j} f_t^{-n}(D_{i,j}^t)$$

such that $h_t \circ f_0 = f_t \circ h_t$.

PROOF. Choose a small compact neighborhood V_t of the attractors of f_t in $D^t = \bigcup_{i,j} D_{i,j}^t$ such that

(i) V_t is a union of finitely many closed intervals whose boundary depends continuously on t;

(ii) $f(V_t) \subset \operatorname{int} V_t$ and V_t is contained in the basin of the attractor;

(iii) the forward orbit of every critical apoint $c_i(t)$ of f_t which is in the basin of an attractor in D^t has a unique pont $d_i(t) \in \operatorname{int} V_t - f(V_t)$ which depends continuously on t.

Let $h_t\colon V_0 - \operatorname{int} f_0(V_0) \to V_t - \operatorname{int} f_t(V_t)$ be a continuous family of homeomorphisms such that $h_t(c_i(0)) = c_i(t)$, $h_t(\partial V_0) = \partial V_t$ and $h_t f_0(x) = f_t h_t(x)$ if $x, f_0(x) \in \partial V_0$. Next extend h_t to V_0 so that $h_t f_0 = f_t h_t$. Next we extend h_t to the backward orbit of V_0. If $x_0 \in M$ is such that $f^n(x_0) \in V_0 - \theta_+(C(f_0))$, let $I(x_0) = \{s \in [0,1]$ exist; $x\colon [0,s] \to M$ continuous such that $f_t^n(x(t)) = h_t f_0^n(x_0)$ and $x(0) = x_0\}$. As in Lemma 2, we have that $I(x_0) = [0,1]$. Define $h_t(x_0) = x(t)$. Next we extend h_t to the backward orbit of the critical points of f_0 as in Lemma 2. It is easy to check that h_t is monotone, $h_t f_0 = f_t h_t$ and h_0 is the identity map.

PROOF OF PROPOSITION 2. Let $f \in \mathcal{U} \cap \mathcal{V}$ and $\mathcal{N}$ be a neighborhood of f contained in $\mathcal{U} \cap \mathcal{V}$. Let us prove that there is a stable map $f_0 \in \mathcal{N}$.

We say that a periodic domain D of f is stable if the periodic points of f in D are hyperbolic. Hence if D is a stable periodic domain of f then every g near f has a stable periodic domain near D. Let ϕ be the map that assigns to each $g \in \mathcal{N}$ the union of the closure of all stable periodic domains of g. Since $\mathcal{N} \subset \mathcal{U}$, $\phi(g)$ is a closed set and ϕ is semicontinuous with respect to the Hausdorff metric. By [**K**,

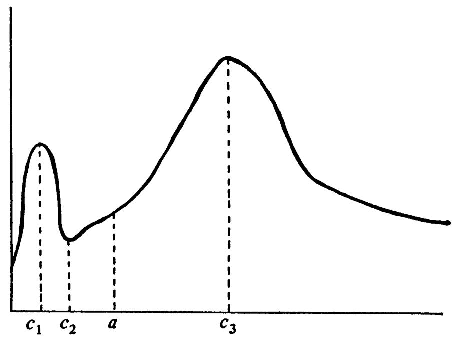

FIGURE 1

p. 71] the set of points of continuity of ϕ is residual. Let $f_0 \in \mathcal{N}$ be a point of continuity of ϕ so that all critical points of f_0 are nondegenerate. Let $\mathcal{N}_1 \subset \mathcal{N}$ be a small connected neighborhood of f_0 such that all periodic domains of $f \in \mathcal{N}_1$ are stable and depend continuously on f. Hence given $f \in \mathcal{N}_1$ there is a continuous curve $f_t \in \mathcal{N}_1$ such that $f_1 = f$. From Lemma 3 and Lemma 1 it follows that f_1 is conjugate to f_0. This proves Proposition 2.

Corollary 1 follows from the density of $\mathcal{V}$ in $\mathcal{S}$. Corollary 2 follows from Corollary 1 and from $\mathcal{S}_0 = \mathcal{F}_0 \cap \mathcal{F}_1$ [**S**, **G**, and **Mi**].

EXAMPLE. Let $f\colon [-1,2] \to [-1,2]$ be a C^∞ map satisfying the following properties:

(1) the restriction of f to $[-\frac{1}{4}, \frac{5}{4}]$ has negative Schwarzian derivative, $f(0) = f(1) = 0$, $f([0,1]) \subset [0,1]$;

(2) f has an attracting fixed points $a \in]-1, -\frac{1}{4}[$ and three critical point c_i, $i = 1, 2, 3$, with $-1 < c_1 < c_2 < a < 0 < c_3 < 1$ as in Figure 1 above;

(3) $f(c_1) \in]0,1[$;

(4) every point in $]1,2[$ is in the basin of a;

(5) if $x \in]-1, 0[$ then either $f^n(x) \in [0,1]$ for some n or x is asymptotic to a;

(6) -1 and c_2 belong to the basin of a.

Notice that if the restriction of f to $[0,1]$ is a stable map in $\mathcal{S}_0$ then there is a neighborhood $\mathcal{N}$ of f such that for any $g \in \mathcal{N}$ there is an interval $[a_0(g), a_1(g)]$ near $[0,1]$ such that $g|[a_0(g), a_1(g)]$ is conjugate to $f|[0,1]$. Also each $g \in \mathcal{N}$ has a critical point $c_1(g)$ near c_1 with $g(c_1(g)) \in [a_0(g), a_1(g)]$. If $f|[0,1] \notin \mathcal{A}$, i.e., if f does not have an attractor in $[0,1]$ then the backward orbit of c_3 is dense in $[0,1]$. The stability of $f|[0,1]$ in $\mathcal{S}_0$ implies that the critical point $c_3(g)$ for $g \in \mathcal{N}$ has its backward orbit dense in $[a_0(g), a_1(g)]$. Therefore no $g \in \mathcal{N}$ would be stable.

To prove Proposition 3 we use two more lemmas.

LEMMA 4. *Let $f\colon [0,1] \to \mathbf{R}$ be a C^∞ function such that $Df(x) > 0$ for $x \in [0, c[$, $Df(x) < 0$ for $x \in]c, 1]$, $D^k f(c) = 0$ for all $k \geq 1$. For each $\lambda > 0$ small enough there exists a C^∞ function $g_\lambda\colon [0,1] \to \mathbf{R}$ such that*

(i) *$g_\lambda(x) = f(x)$ if $x \notin [c + \frac{1}{4}\lambda, c + 4\lambda]$,*

(ii) *$Dg_\lambda(x) < 0$ if $x > 0$ and $x \notin [c + \frac{1}{2}\lambda, c + 2\lambda]$,*

(iii) *$Dg_\lambda(x) = 0$ for all $x \in [c + \frac{1}{2}\lambda, c + 2\lambda]$,*

(iv) *Given $n \in \mathbf{N}$ and $\varepsilon > 0$ there exist $\delta > 0$ such that if $\lambda < \delta$ then $\|g_\lambda - f\|_n < \varepsilon$. Here $\|g - f\|_n = \sup\{|D^k g(x) - D^k f(x)|;\ 0 \leq k \leq n$ and $x \in [0,1]\}$.*

PROOF. Let $\alpha\colon \mathbf{R} \to \mathbf{R}$ be a C^∞ function such that $\alpha(\mathbf{R}) \subset [0,1]$, $\alpha(x) = 0$ if $|x| \geq 2\operatorname{Log} 2$ and $\alpha(x) = 1$ if $|x| \leq \operatorname{Log} 2$. Let $\beta_\lambda\colon \mathbf{R} \to \mathbf{R}$ be such that $\beta_\lambda(x) = 0$ if $x \notin [c + \frac{1}{4}\lambda, c + 4\lambda]$ and $\beta_\lambda(x) = \alpha(\operatorname{Log} x/\lambda)$ if $x \in]c + \frac{1}{4}\lambda, c + 4\lambda[$. Define $g_\lambda\colon [0,1] \to \mathbf{R}$ as $g_\lambda(x) = f(x) + \beta_\lambda(x)\cdot(f(\lambda) - f(x))$. Since all derivatives of f vanish at c there exists $\delta_r > 0$ such that if $\lambda < \delta_r$ then

(1) $|Dg^k(x)| < \lambda^{r+1}$ for all $x \in [c, c + 2\lambda]$ and all $k = 1, \ldots, r$. On the other hand there exists $C_r > 0$ such that

(2) $|D^j\beta_\lambda(x)| < C_r/\lambda^r$ for all $k = 1, \ldots, r$. The lemma follows from (1) and (2).

LEMMA 5. *Let $f \in \mathcal{F}_0 \cap \mathcal{F}_1$ be such that $J(f)$ contains a unique critical point $c(f)$ which is nonwandering and eventually periodic. Then* $\inf\{|p - c|/|f^k(p) - c|$; *$p$ is a periodic point of f and k is an integer such that $f^k(p) \neq p\} = 0$.*

PROOF. Let I be a small interval containing c in its interior such that I is disjoint from the forward orbit of c. Since all other critical points are asymptotic to periodic points in periodic domains we may assume that the forward orbit $\theta_+(x)$ of $x \in I$ does not intersect I if it contains another critical point of f. Let $U \subset f(I)$ be the domain of the first return map to I and let $\phi\colon U \to I$ be the first return map. The restriction of ϕ to each connected component of U is a monotone map onto I. Hence each connected component of U has a fixed point of ϕ. This is a periodic point of f whose orbit contains a unique point in I. Since $c \notin U$ but is in the boundary of U we have that c is assumulated by connected components of U. This proves the lemma.

PROOF OF PROPOSITION 3. (a) Let $f \in \mathcal{U} \cap \mathcal{V}$ be such that f has a unique critical point $c(f) \in J(\lambda)$, $c(f)$ is nonwandering and eventually periodic but it is not in the boundary of a periodic domain of f. All other critical points of f are asymptotic to some point in a periodic domain. Let $[a,b] \subset M$ be a small interval such that $[a,b]$ is disjoint from the forward orbits of all critical points. Assume that $[a,b]$ is also disjoint from all periodic domains of f. Let f_t be a continuous one-parameter family of maps such that $f_1 = f$, $f_t = f$ in $M - [a,b]$, f_t has a unique critical point at $c(f)$, $c(f)$ is a flat critical point for f_0. Let $\mathcal{S} \subset \operatorname{End}(m)$ be the set of maps g such that $g = f$ in $M - [a,b]$, g has a unique critical point $c = c(f)$ in $[a,b]$ and is flat at this critical point. Clearly $\mathcal{S}$ is a closed connected subset of $\operatorname{End}(M)$. By Lemmas 1 and 2 we get that each $g \in \mathcal{S}$ is semiconjugate to f (the semiconjugacy can be defined to be the identity map in the periodic domains of f and in the critical set $C(f) = C(g)$, extended to the backward orbit of these sets by Lemma 2 and finally extended to M using Lemma 1). Next we show that the set $\mathcal{S}_\infty$ of maps in $\mathcal{S}$ having infinitely many periodic domains is dense. Let $\mathcal{S}_k$ be the set of maps in $\mathcal{S}$ having at least k hyperbolic attractors in disjoint periodic domains. Clearly $\mathcal{S}_k$ is open and $\mathcal{S}_k \subset \mathcal{S}_{k-1}$. From Lemmas 4 and 5 it follows that any map $g \in \mathcal{S}_{k-1}$ can be approximated by another map $g \in \mathcal{S}$ having an extra attractor of higher period. Hence $g \in \mathcal{S}_k$. Therefore $\mathcal{S}_k$ is open and dense. Since $\mathcal{S}$ is a Baire space it follows that $\bigcap_{k=1}^{\infty} \mathcal{S}_k$ is dense in $\mathcal{S}$.

(b) Let $\phi\colon U \subset [a,b] \to [a,b]$ be the first return map. Since $c(f)$ is not in U but it is in the boundary of U we have that c is accumulated by connected components of U and the restriction of ϕ to each component is onto. Hence there is a sequence $x_i \to c$ such that $\phi(x_i) = c$. Let $y_i = f(x_i)$ and I_i be the f-image of the connected component of U which contains x_i. Thus each I_i is an open interval containing y_i

and there is an integer n_i such that $f^j(I_i) \cap [a,b] = \emptyset$ for $0 \le j < n_i$, $f^{n_i}(I_i) = [a,b]$. Furthermore, for any integers i, j we have that

$$(*) \qquad f^k(I_i) \cap f^l(I_j) = \emptyset \quad \text{if } 0 \le k < n_i,\ 0 \le l < n_j.$$

For each $\lambda > 0$ sufficiently small let S_λ denote the set of C^∞ maps g such that $g = f$ in $M - [a,b]$, $g(x) = f(c)$ if $|x - c| \le \lambda$ and $Dg(x) \ne 0$ if $x \in [a,b]$ and $|x - c| > \lambda$. Construct sequences ε_n, $i_n \in \mathbf{N}$, and g_n such that

$$\begin{aligned}
&g_{n+1} \in S_{\varepsilon_{n+1}}, \quad \varepsilon_{n+1} < \tfrac{1}{4}\varepsilon_n, \qquad i_{n+1} > i_n,\\
&g_{n+1}(x) = g_n(x) \quad \text{if } |x - c| \ge \varepsilon_{n-1},\\
&|g_{n+1} - g_n|_n < 1/2^n,\\
&g_{n+1}(\{x; \tfrac{1}{2}\varepsilon_n < x - c < \varepsilon_n\}) \subset I_{i_{n+1}}(\varepsilon_{n+1}),
\end{aligned}$$

where

$$I_i(\varepsilon) = \{x \in I_i; \tfrac{1}{2}\varepsilon < f^{n_i}(x) - c < \varepsilon\}.$$

Clearly g_n converges to a C^∞ function g having a unique critical point in $[a,b]$. Furthermore

$$g(\{x; \tfrac{1}{2}\varepsilon_n < x - c < \varepsilon_n\}) \subset I_{i_{n+1}}(\varepsilon_{n+1}) \quad \text{for every } n.$$

If $A = I_{i_1}(\varepsilon_1)$ we have that $g^{j_n}(A) = I_{j_n}(\varepsilon_n)$ for some j_n and the intervals $g^j(A)$ are pairwise disjoint. Since A does not intersect the backward orbits of the critical points of g it follows that there is a wandering domain of g containing A. This finishes the proof of Proposition 3.

REFERENCES

[**Su**] D. Sullivan, *Quasiconformal homeomorphisms and dynamics.* I: *a solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. (2) **122** (1985), 401–418.

[**MSS**] R. Mañe, P. Sad and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. (4) **16** (1983), 193–217.

[**P.**] Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **47** (1919), 161–271; **48** (1920), 33–94; 203–314.

[**S**] D. Singer, *Stable orbits and bifurcations of maps of the interval*, SIAM J. Appl. Math. **35** (1978), 260.

[**Mi**] M. Misiurewicz, *Absolutely continuous measures for certain maps of the interval*, Publ. Math. Inst. Hautes Études Sci. **53** (1981), pp. 17–52.

[**G**] J. Guckenheimer, *Sensitive dependence on initial conditions for one dimensional maps*, Comm. Math. Phys. **70** (1979), 133–160.

[**Ma**] R. Mañe, *Hyperbolicity, sinks, and measure in one dimensional dynamics*, Comm. Math. Phys. **100** (1985), 495–524.

[**Y**] L. S. Young, *A closing lemma on the interval*, Invent. Math. **54** (1979), 179–187.

[**K**] K. Kuratowski, *Topology.* II, Academic Press, New York, 1968.

[**H**] C. R. Hall, *A* C^∞ *Denjoy counterexample*, Ergodic Theory and Dynamical Systems **1** (1981), 261–272.

[**S-I**] A. N. Sharkowskii and A. F. Ivanov, C^∞*-mappings of an interval with attracting cycles of arbitrarily large periods*, Ukrain Mat. Zh. **35** (1983).

[**dM-St**] W. De Melo and S. Van Strien, *A structure theorem in one dimensional dynamics*, preprint IMPA.

INSTITUTO DE MATEMÁTICA PURA E APLICADA, ESTRADA DONA CATORINA 110, CEP 22460, RIO DE JANEIRO, RJ, BRAZIL