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A GENERALIZATION OF SMITH THEORY

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ABSTRACT. Using Bredon cohomology, new relations are obtained between the mod p Betti numbers of a finite G-CW complex and its singular subspace, where G is a finite p-group.

Let G be a p-group of finite order p^e and let X be a finite dimensional G-CW complex such that $H^*(X)$ is finite dimensional, where cohomology is understood with mod p coefficients. Let SX denote the subcomplex of singular points of X, that is, of points fixed by some $g \neq e$. Finally, let $FX = X/SX$; FX is a based G-CW complex such that the action off the basepoint is free. We seek relations among the mod p Betti numbers

$$a_q = \dim \tilde{H}^q(FX/G), \quad b_q = \dim H^q(X), \quad \text{and} \quad c_q = \dim H^q(SX).$$

If G is cyclic of order p, so that $SX = X^G$, Floyd's formulation [2, 4.4] of Smith theory gives the following inequality for $q \geq 0$ and $r \geq 0$, where r is odd if p is odd:

$$(*) \qquad a_q + c_q + c_{q+1} + \cdots + c_{q+r} \leq b_q + b_{q+1} + \cdots + b_{q+r} + a_{q+r+1}.$$

Floyd [2, p. 146] also gives the Euler characteristic relation

$$\chi(X) = \chi(X^G) + p\tilde{\chi}(FX/G),$$

where the reduced Euler characteristic of a based space is one less than the actual Euler characteristic. With $q = 0$ and r large, $(*)$ gives $\sum c_q \leq \sum b_q$. When X is a mod p cohomology sphere, the last inequality and the relation $\chi(X) \equiv \chi(X^G)$ mod p immediately imply Smith's conclusion that X^G is also a mod p cohomology sphere.

In the general case, classical Smith theory and induction on e imply dimensional restrictions on the cohomology of all fixed point spaces X^H and therefore, by inductive use of Mayer-Vietoris sequences, on the cohomology of SX. Our new observation is that much sharper dimensional restrictions can be derived directly.

THEOREM. *The following inequality holds for any $q \geq 0$ and $r \geq 0$:*

$$a_q + \sum_{i=0}^{r} (p^e - 1)^i c_{q+i} \leq \sum_{i=0}^{r} (p^e - 1)^i b_{q+i} + (p^e - 1)^{r+1} a_{q+r+1}.$$

In particular, with r large,

$$\sum_{i \geq 0} (p^e - 1)^i c_{q+i} \leq \sum_{i \geq 0} (p^e - 1)^i b_{q+i}.$$

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Moreover,

$$\chi(X) = \chi(SX) + p^e \tilde{\chi}(FX/G).$$

Since $a_q = 0$ for q large by the finite dimensionality of X, the inequalities and the finiteness of the b_q imply the finiteness of the a_q and c_q. Of course, the Euler characteristic formula is trivial when X is a finite G-CW complex. If $p = 2$, the inequalities in the classical case $e = 1$ are those given by Floyd. If $p > 2$, the inequalities in the case $e = 1$ differ from those of Floyd due to the coefficients $(p-1)^i$. We shall explain after the proof why the cyclic groups of odd prime order behave exceptionally.

When G is cyclic, our inequalities do not appear to give new information; in the noncyclic case, they do. For example, our inequalities obviously imply that if $c_q = b_q$ for all $q > n$, then $c_n \leq b_n$. Even this simple fact does not seem to follow from any previous version of Smith theory.

To prove the theorem, observe first that the inequalities for $r > 0$ will follow inductively from those for $r = 0$, which read

$$(\#) \qquad a_q + c_q \leq b_q + (p^e - 1)a_{q+1}.$$

To obtain the inequalities for $r = 1$, we add $(p^e - 1)c_{q+1}$ to both sides, and so on.

The proof of the theorem is an application of Bredon cohomology [**1**]. Let $G\mathcal{O}$ denote the category of orbits G/H and G-maps between them. A coefficient system is a contravariant functor from $G\mathcal{O}$ to the category of Abelian groups. For each coefficient system M, there is a cohomology theory $H_G^*(?; M)$ on G-CW complexes. It is characterized by a dimension axiom: when restricted to the category $G\mathcal{O}$, $H_G^q(?; M)$ is the coefficient system M if $q = 0$ and is identically zero if $q \neq 0$. An exact sequence of coefficient systems $0 \to L \to M \to N \to 0$ gives rise to a natural long exact sequence

$$\cdots \to H_G^q(X; L) \to H_G^q(X; M) \to H_G^q(X; N) \to H_G^{q+1}(X; L) \to \cdots.$$

There are coefficient systems L, M, and N such that

$$H_G^q(X; L) \cong \tilde{H}^q(FX/G), \quad H_G^q(X; M) \cong H^q(X), \quad \text{and} \quad H_G^q(X; N) \cong H^q(SX).$$

In fact, to specify L, M, and N, we can and must set

$$L(?) = \tilde{H}^0(F?/G), \quad M(?) = H^0(?), \quad \text{and} \quad N(?) = H^0(S?)$$

on orbits and on G-maps between orbits. Thus $L(G) = Z_p$ and $L(G/H) = 0$ if $H \neq e$; $M(G/H) = Z_p[G/H]$ for all H; and $N(G) = 0$ and $N(G/H) = Z_p[G/H]$ if $H \neq e$. In particular, $M(G)$ is the group ring $Z_p[G]$ regarded as a G-module.

Let I be the augmentation ideal in $Z_p[G]$, let s be maximal such that $I^s \neq 0$, and let d_n be the dimension of the Z_p-vector space I^n/I^{n+1} for $1 \leq n \leq s$. The values of the d_n are given by Jennings' formula [**3**, 2.10], but we only need the relations $d_s = 1$ and $\sum d_n = p^e - 1$. Write I^n ambiguously for both the ideal and the coefficient system with $I^n(G) = I^n$ and $I^n(G/H) = 0$ for $H \neq e$. Then I is a subcoefficient system of M, and $M/I = L \oplus N$ since $Z_p[G]/I \cong Z_p$ and since the map

$$Z_p[G/H] = H^0(G/H) \to H^0(G) = Z_p[G]$$

induced by a G-map $G \to G/H$ with $H \neq G$ takes values in I. The long exact cohomology sequence associated to the short exact sequence

$$0 \to I \to M \to L \oplus N \to 0$$

gives the inequality

$$a_q + c_q \leq b_q + \dim H_G^{q+1}(X; I)$$

and the Euler characteristic formula

$$\chi(X) = \chi(SX) + \chi(FX/G) + \chi(H_G^*(X; I)).$$

For $1 \leq n < s$, we have an evident short exact sequence

$$0 \to I^{n+1} \to I^n \to d_n L \to 0,$$

where dL denotes the direct sum of d copies of L. The resulting long exact sequence in cohomology gives

$$\dim H_G^q(X; I^n) \leq d_n a_q + \dim H_G^q(X; I^{n+1})$$

and

$$\chi(H_G^*(X; I^n)) = d_n \tilde{\chi}(FX/G) + \chi(H_G^*(X; I^{n+1})).$$

Since $d_s = 1$, $I^s = L$ and $H_G^q(X; I^s) = \tilde{H}^q(FX/G)$. Our theorem follows.

If G is cyclic of odd prime order p, then $s = p - 1$ and $I^{p-1} = L$. Here $Z_p[G]/I^{p-1} \cong I$ as $Z_p[G]$-modules. If t generates G, then the norm $\sum t^i$ generates I^{p-1}, and it follows that $M/I^{p-1} \cong I \oplus N$ as coefficient systems. We thus have short exact sequences

$$0 \to I \to M \to L \oplus N \to 0 \quad \text{and} \quad 0 \to L \to M \to I \oplus N \to 0.$$

With $\bar{a}_q = \dim H_G^q(X; I)$, these imply the two inequalities

$$a_q + c_q \leq b_q + \bar{a}_{q+1} \quad \text{and} \quad \bar{a}_q + c_q \leq b_q + a_{q+1}.$$

Floyd's inequalities (*) follow inductively.

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