PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 4, December 1987

PARTITIONING TOPOLOGICAL SPACES INTO COUNTABLY MANY PIECES

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(Communicated by Dennis Burke)

ABSTRACT. We assume $X \to (\mathrm{top}\,\omega+1)^1_\omega$ and determine which larger α can replace $\omega+1$. If X is first countable, any countable α can replace $\omega+1$. If the character of X is ω_1, it is consistent and independent whether ω^2+1 can always replace $\omega+1$. Consistently ω_1 cannot replace $\omega+1$ for any X of size ω_1.

There have recently been several results regarding partitions of topological spaces (for a survey, see [**2**]). We wish to add some results to this literature. In particular, we partition a (regular) topological space into countably many pieces and consider conditions under which one piece must contain a homeomorphic copy of a countable ordinal as a subspace. We write

$$X \to (\mathrm{top}\,\alpha)^1_\omega$$

to mean that for each partition $f\colon X \to \omega$ there is a homeomorphic embedding h from the ordinal α with its usual order topology into X such that f is constant on the range of h. As usual, $\not\to$ denotes the negation of this relation.

We consider spaces which satisfy the relation

$$X \to (\mathrm{top}\,\omega+1)^1_\omega.$$

For example, note that for first countable spaces, this relation just says that X is not σ-discrete. In this article we wish to determine, for X satisfying the above relation, which other ordinals can automatically replace $\omega+1$. Our first theorem uses the cardinal b which is the smallest cardinality of an unbounded family of functions in ${}^\omega\omega$. Terminology is from [**1**].

THEOREM 1. *Suppose the character of X at each point is less than b and $X \to (\mathrm{top}\,\omega+1)^1_\omega$. Then for each $\alpha<\omega_1$ we have $X \to (\mathrm{top}\,\alpha)^1_\omega$.*

PROOF. Let X be as in the hypothesis of the theorem and $X=\bigcup\{X_n\colon n<\omega\}$ be a partition. We will show that there is some X_n which contains a homeomorphic copy of each countable ordinal. First, we construct the Cantor-Bendixson derivatives of X, and calculate the Cantor-Bendixson height. By the hypothesis on X, there are only two possibilities:

(i) the height of X_n is uncountable for some n;

(ii) for some n, X_n contains a dense-in-itself subspace Y.

Received by the editors June 23, 1986 and, in revised form, September 23, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54A25.

Key words and phrases. Partition, diamond, countable ordinal.

Research partially supported by NSERC grant no. A3140. The authors would like to acknowledge the hospitality of the University of Saskatchewan and the Mathematical Institute of the Hungarian Academy of Sciences where this work was produced.

In case (i) we shall prove by induction that if x is in the αth level of X_n, then there is a homeomorphic copy H of α contained in the first α levels of X_n such that $H \cup \{x\}$ is homeomorphic to $\alpha + 1$. In case (ii) we shall prove by induction on α that if $x \in Y$ there is a homeomorphic copy H of α in Y such that $H \cup \{x\}$ is homeomorphic to $\alpha + 1$. Fortunately, the proofs can be given together.

First note that it suffices to do the inductive step only for indecomposable ordinals, i.e. those of the form $\alpha = \omega^\beta$.

Given $\alpha = \omega^\beta$, we can find limit ordinals α_n such that $\sum \alpha_n = \alpha$. Now, for each relevant $x \in X_n$ we can find a sequence $\langle x_m \rangle$ converging to x, such that for each m there is a sequence $\langle s_m(\eta) \rangle$ of type α_m such that

$$\{s_m(\eta) \colon \eta < \alpha_m\} \cup \{x_m\}$$

is homeomorphic to $\alpha_m + 1$, and contained in X_n.

We now claim that for each m there exists $\beta_m < \alpha_m$ such that

$$\bigcup_{m<\omega} \{s_m(\eta) \colon \beta_m < \eta < \alpha_n\} \cup \{x_m \colon m < \omega\} \cup \{x\}$$

is homeomorphic to $\alpha + 1$. By the indecomposability of each α_m, it only remains to show that we can find the β_m such that each neighborhood of x contains

$$\bigcup_{k<m} \{s_m(\eta) \colon \beta_m < \eta < \alpha_n\} \cup \{x_m \colon k < m\} \cup \{x\}$$

for some $k < \omega$.

In order to prove this, we first find $\lambda < b$ and $\{U_j \colon \gamma < \lambda\}$ a local base for X at x. For each m, pick a cofinal sequence η_j in α_m of order type ω. For each $\gamma < \lambda$ define $f_\gamma(m)$ to be the least j such that

$$\{s_m(\eta) \colon \eta_j < \eta < \alpha_m\} \subseteq U_\gamma.$$

For those finitely many m for which no such j exists, let $f_\gamma(m) = 0$. Obtain $f \colon \omega \to \omega$ dominating each f_γ and let $\beta_m = f(m)$. It is now straightforward to complete the proof.

COROLLARY 2. *If X is first countable and not σ-discrete, then $X \to (\operatorname{top} \alpha)^1_\omega$ for each $\alpha < \omega_1$.*

COROLLARY 3. *Assume* MA(ω_1). *If X has character ω_1 and satisfies $X \to (\operatorname{top} \alpha)^1_\omega$ for $\alpha = \omega + 1$, then it satisfies the relation for each $\alpha < \omega_1$.*

We will now demonstrate that this latter corollary does indeed need an additional assumption outside ZFC.

THEOREM 4. *Assume $\diamond$. There is a space X of character ω_1 such that $X \to (\operatorname{top} \omega + 1)^1_\omega$ while $X \not\to (\operatorname{top} \omega^2 + 1)^1_\omega$.*

PROOF. We use the following equivalent form of $\diamond$. There is a sequence $\langle f_\alpha \colon \alpha < \omega_1 \rangle$ with $f_\alpha \colon \alpha \to \omega$ such that for every $f \colon \omega_1 \to \omega$ the set $\{\alpha < \omega_1 \colon f_\alpha = f | \alpha\}$ is stationary. We fix a sequence with this property.

We will construct, by recursion on $\alpha < \omega_1$, a topology $\mathcal{T}$ on ω_1 such that it has character ω_1, satisfies $\langle \omega_1, \mathcal{T} \rangle \to (\operatorname{top} \omega + 1)^1_\omega$ but does not even contain a homeomorphic copy of ω^2+1. We construct a sequence $\langle \mathcal{T}_\alpha : \alpha < \omega_1 \rangle$ of topologies such that at stage α:

(i) $\mathcal{T}_\alpha$ is a zero-dimensional topology on α refining the order topology on α;

(ii) for any $\beta < \alpha$, $\mathcal{T}_\beta = \mathcal{T}_\alpha \cap P(\beta)$.

For successors of successors, ordinals of the form $\alpha+2$, we let $\mathcal{T}_{\alpha+2}$ be generated by $\mathcal{T}_{\alpha+1}$ and $\{\alpha+1\}$. For limit ordinals α we let $\mathcal{T}_\alpha$ be generated by $\bigcup\{\mathcal{T}_\beta : \beta < \alpha\}$.

For ordinals of the form $\alpha+1$ where α is a limit ordinal, we construct a sequence $\{t_k\}$ of type ω, converging to α with the following additional property: if $f_\alpha^{-1}(\{i\})$ is unbounded in α for $i < \omega$, then $f_\alpha(t_k) = i$ holds for infinitely many k. This can obviously be done. For each k, let $a(k,j)$ be an increasing sequence of successors cofinal in t_k; we allow $a(k,j) = t_k$ for successor t_k.

Now, for each $n < \omega$ and each $g : \omega \backslash n \to \omega$ define $U(n,g)$ to be

$$\{\alpha\} \cup \{\eta : a(k, g(k)) \le \eta \le t_k \text{ and } k \ge n\}.$$

We let $\mathcal{T}_{\alpha+1}$ be generated by

$$\mathcal{T}_\alpha \cup \{U(n,g) : n < \omega \text{ and } g : \omega \backslash n \to \omega\}.$$

It is straightforward to check that (i) and (ii) are still satisfied.

We let $\mathcal{T} = \mathcal{T}_{\omega_1}$. It is easy to check that $\langle \omega_1, \mathcal{T} \rangle$ is zero-dimensional and Hausdorff. Note that CH ensures that the character is ω_1. To see that the space does not contain a homeomorphic copy of ω^2+1, it suffices to note the following. If α is a limit ordinal and $Y \subseteq \alpha$ is such that $Y \cup \{\alpha\}$ is compact, then there is some $\beta < \alpha$ such that

$$Y \cap (\alpha \backslash \beta) \subseteq \{t_k : k < \omega\} \cap (\alpha \backslash \beta).$$

It remains to prove the positive partition relation for $X = \langle \omega_1, \mathcal{T} \rangle$. Suppose $f : \omega_1 \to \omega$ is a partition of X. Call $i < \omega$ *small* if $f^{-1}(\{i\})$ is bounded, otherwise i is *large*. Notice that there is a $\beta < \omega_1$ such that for every small i, $f^{-1}(\{i\}) \subseteq \beta$, and there is a closed unbounded set C such that, if $i < \omega$ is large, then $f^{-1}(\{i\})$ is unbounded in α for every $\alpha \in C$.

By the diamond property, there is an $\alpha \in C$, $\alpha > \beta$ with $f_\alpha = f \mid \alpha$. As $\alpha > \beta$, $f(\alpha)$ cannot be small. If $f(\alpha) = i$, i is large, so

$$\{t_k : f_\alpha(t_k) = f(t_k) = i\} \cup \{\alpha\}$$

is an i-colored $\omega+1$.

It is not hard to show that Theorem 4 is best possible in the following sense:

$$\text{if } X \to (\operatorname{top} \omega + 1)^1 \text{ then } X \to (\operatorname{top} \omega^2)^1_\omega.$$

Indeed, if β is homeomorphic to countably many disjoint copies of α, then

$$X \to (\operatorname{top} \alpha)^1_\omega \quad \text{implies} \quad X \to (\operatorname{top} \beta)^1_\omega.$$

It may be interesting which other implications hold. The following is relevant.

THEOREM 5. *Assume $\diamond$. For any space X of cardinality ω_1 we have*

$$X \not\to (\operatorname{top} \omega_1)^1_\omega.$$

In fact X can be partitioned into uncountably many pieces such that each homeomorphic copy of ω_1 intersets each piece.

PROOF. We partition X by coloring it with uncountably many colors. Suppose the underlying set of X is ω_1. We will color ω_1 in stages; we let C_α denote the set of ordinals colored after α stages.

Let $\langle S_\alpha : \alpha < \omega_1 \rangle$ witness $\diamond$ where each $S_\alpha \subseteq \alpha$. At stage α we suppose that $\alpha \subseteq C_\alpha$. If some $\beta \geq \alpha$ is in the closure of S_α, we color β the first color which does not occur in S_α. Also, if α has not yet been colored we color it arbitrarily.

Now suppose $S \subseteq \omega_1$, which is homeomorphic to ω_1. Since S is locally countable, $\{\alpha < \omega_1 : S \cap \alpha \text{ is open}\}$ is a cub. For each γ which is a limit of this cub we have $S \cap \gamma$ is not compact, and hence $S \cap \gamma$ is not closed. Also, $\{\alpha : C_\alpha = \alpha\}$ is cub. Therefore by $\diamond$ there is some stationary set E such that for each $\alpha \in E$, $S_\alpha = S \cap \alpha$, $\alpha = C_\alpha$, and $S \cap \alpha$ is not closed. Hence the construction gives for each $\alpha \in E$ some β_α which is colored the first color not appearing in S_α. Furthermore, since S contains the closure of each of its countable subsets, each $\beta_\alpha \in S$. Thus $S = \bigcup\{S_\alpha : \alpha \in E\}$ must have all colors.

Theorem 5 has extensions recently proved by others. J. Steprans has shown the relative consistency of $X \not\to (\text{top}\,\omega_1)^1_2$ for $X = \{0,1\}^{\omega_1}$ and J. Merrill has shown the relative consistency of this relation for all X of size ω_2. It was already known to J. Silver that the relation was relatively consistent for the ordinal space ω_2. S. Shelah has proven the independence of this latter result. However, the following problems are still unsolved in ZFC.

1. Is it true that for every regular space of size ω_1 we have $X \not\to (\text{top}\,\omega_1)^1_2$? A result with ω replacing 2 would still be interesting.

2. Is there a regular space X such that $X \to (\text{top}\,\omega + 1)^1_\omega$ but $X \not\to (\text{top}\,\alpha)^1_\omega$ for some countable $\alpha > \omega^2$?

3. Is there a regular space X such that $X \to (\text{top}\,\omega^2 + 1)^1_\omega$ but $X \not\to (\text{top}\,\alpha)^1_\omega$ for some countable $\alpha > \omega^3$?

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