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A UNIMODALITY RESULT IN THE ENUMERATION OF SUBGROUPS OF A FINITE ABELIAN GROUP

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ABSTRACT. The number of subgroups of order p^k in an abelian group G of order p^n is a polynomial in p, $\alpha_\lambda(k;p)$, determined by the type λ of G. It is well known that $\alpha_\lambda(k;p) = \alpha_\lambda(n-k;p)$. Using a recent result from the theory of Hall-Littlewood symmetric functions, we prove that $\alpha_\lambda(k;p)$, $0 \le k \le n$, is a unimodal sequence of polynomials. That is, for $1 \le k \le n/2$, $\alpha_\lambda(k;p) - \alpha_\lambda(k-1;p)$ is a polynomial in p with nonnegative coefficients.

1. Finite abelian p-groups. If G is an abelian group of order p^n, p a prime, then G is isomorphic to a direct product of cyclic groups

$$\mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \mathbf{Z}/p^{\lambda_2}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_l}\mathbf{Z}.$$

The partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l, 0, \ldots)$ where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l$ is called the *type* of G. We write $\lambda \vdash n$ since $\sum \lambda_i = n$.

If H is a subgroup of G and the type of H is ν, then $\nu_i \le \lambda_i$ for all i. The number of subgroups of type ν is (using, e.g., [**2**, 8.1])

$$\prod_{i\ge 1} p^{\nu'_{i+1}(\lambda'_i - \nu'_i)} \begin{bmatrix} \lambda'_i - \nu'_{i+1} \\ \nu'_i - \nu'_{i+1} \end{bmatrix}_p \tag{1}$$

where $\lambda'_i = \dim_{\mathbf{Z}/p\mathbf{Z}}(p^{i-1}G/p^iG)$, $\nu'_i = \dim_{\mathbf{Z}/p\mathbf{Z}}(p^{i-1}H/p^iH)$ and $[{n \atop k}]_p$ is the number of k-dimensional subspaces of an n-dimensional vector space over $\mathbf{Z}/p\mathbf{Z}$. Combinatorially, λ' and ν' are the conjugate partitions of λ and ν, respectively, and $[{n \atop k}]_p$ is the p-binomial coefficient (see, e.g., [**1**, p. 81 and p. 78]).

It follows immediately from (1) that the number of subgroups of order p^k in a finite abelian p-group of type λ, $\alpha_\lambda(k;p)$, is a polynomial in p with nonnegative (integer) coefficients. It is well known (see, e.g., [**10**, p. 87]) that $\alpha_\lambda(k;p) = \alpha_\lambda(n-k;p)$. In fact, if $g^\lambda_{\mu\nu}(p)$ is the number of subgroups H of type ν in a finite abelian p-group G of type λ such that G/H is of type μ, then P. Hall (see [**10**, p. 93]) showed:

(i) $g^\lambda_{\mu\nu}(p)$ is a polynomial in p with integer coefficients;

(ii) $g^\lambda_{\mu\nu}(p) = g^\lambda_{\nu\mu}(p)$;

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(iii) If $c^\lambda_{\mu\nu}$ is the Littlewood-Richardson coefficient [**10**, p. 68],

$$c^\lambda_{\mu\nu} = 0 \Rightarrow g^\lambda_{\mu\nu}(p) = 0;$$
$$c^\lambda_{\mu\nu} \neq 0 \Rightarrow g^\lambda_{\mu\nu}(p) \text{ has degree } n(\lambda)-n(\mu)-n(\nu) \text{ and leading coefficient } c^\lambda_{\mu\nu}, \text{ where } n(\lambda) = \textstyle\sum(i-1)\lambda_i.$$

The aim of this paper is to prove the following result:

THEOREM. *Let $\lambda \vdash n$ and $1 \leq k \leq n/2$. Then $\alpha_\lambda(k;p) - \alpha_\lambda(k-1;p)$ has nonnegative coefficients.*

Thus, as conjectured by, e.g., P. Massell and M. Gikas, for fixed $\lambda \vdash n$ and p the sequence $\alpha_\lambda(0;p), \alpha_\lambda(1;p), \ldots, \alpha_\lambda(n;p)$ is unimodal.

2. Hall-Littlewood symmetric functions. P. Hall (see [**10**, p. 112]) indirectly defined symmetric functions $P_\lambda(x;t)$, $x = (x_1, x_2, \ldots)$, with the property that

$$P_\mu(x;t)P_\nu(x;t) = \sum_\lambda g^\lambda_{\mu\nu}(t^{-1})t^{n(\lambda)-n(\mu)-n(\nu)}P_\lambda(x;t). \tag{2}$$

D. E. Littlewood [**9**] gave a direct definition of these symmetric functions, called Hall-Littlewood symmetric functions. Macdonald [**10**, Chapter 3] derives the following facts from Littlewood's expression. First, if Λ is the ring of symmetric functions in $x = (x_1, x_2, \ldots)$ [**10**, p. 10], then $\{P_\lambda(x;t)\}_\lambda$ is a $\mathbf{Z}[t]$-basis of $\Lambda \otimes \mathbf{Z}[t]$ [**10**, p. 105]. Second, if $h_n(x) = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}$ is the homogeneous symmetric function, then

$$h_n(x) = \sum_{\lambda \vdash n} t^{n(\lambda)}P_\lambda(x;t) \tag{3}$$

(see [**10**, p. 117]).

Finally, Macdonald [**10**, p. 129] states Lascoux and Schützenberger's result [**8**] that when the Schur function s_ρ is expanded in terms of Hall-Littlewood symmetric functions

$$s_\rho(x) = \sum_\lambda K_{\rho,\lambda}(t)P_\lambda(x;t)$$

the polynomials $K_{\rho,\lambda}(t)$ have nonnegative coefficients. In [**5**, Chapter 3] we discuss and complete Lascoux and Schützenberger's combinatorial proof [**12**].

3. Proof of the main theorem. We desire to show that if $\lambda \vdash n$ and $1 \leq k \leq n/2$, then $\alpha_\lambda(k;p) - \alpha_\lambda(k-1;p)$ has nonnegative coefficients. It follows from (2) and (3) that the polynomial $\alpha_\lambda(k;t^{-1})t^{n(\lambda)}$ is the coefficient of $P_\lambda(x;t)$ when the homogeneous symmetric function $h_{(n-k,k)}(x)$ is expanded in terms of the Hall-Littlewood symmetric functions. So,

$$h_{(n-k,k)}(x) - h_{(n-k+1,k-1)}(x) = \sum_{\lambda \vdash n} \left(\alpha_\lambda(k;t^{-1}) - \alpha_\lambda(k-1;t^{-1})\right) t^{n(\lambda)}P_\lambda(x;t).$$

On the other hand, the Jacobi-Trudi identity (see, e.g., [**10**, p. 25]) asserts that $s_\rho = \det|h_{\rho_i - i + j}|$. Hence

$$h_{(n-k,k)}(x) - h_{(n-k+1,k-1)}(x) = s_{(n-k,k)}(x).$$

Since the $P_\lambda(x;t)$ are linearly independent, comparing the two equations above yields

$$\left(\alpha_\lambda(k;t^{-1}) - \alpha_\lambda(k-1;t^{-1})\right) t^{n(\lambda)} = K_{(n-k,k),\lambda}(t).$$

The result of Lascoux and Schützenberger now is seen to imply that $\alpha_\lambda(k;p) - \alpha_\lambda(k-1;p)$ has nonnegative coefficients.

NOTE. Using, e.g., [**10**, pp. 105, 130–131], we find that for $1 \le k \le n/2$,

$$\begin{aligned} \alpha_\lambda(k;p) &= \alpha_\lambda(k-1;p) && \text{if } k > \sum_{i\ge 2}\lambda_i, \\ \alpha_\lambda(k;p) &\equiv \alpha_\lambda(k-1;p) + p^k \pmod{p^{k+1}} && \text{otherwise.} \end{aligned}$$

4. Chains of subgroups. Let $\lambda \vdash n$ and $S = \{a_1, a_2, \dots, a_j\} \subseteq \{1, 2, \dots, n-1\}$ where $a_1 < a_2 < \cdots < a_j$. Define $\alpha_\lambda(S;p)$ to be the number of chains of subgroups

$$\{e\} \subset H_1 \subset H_2 \subset \cdots \subset H_j \subset G$$

in a finite abelian p-group G of type λ, where H_i has order p^{a_i}. By (1), $\alpha_\lambda(S;p)$ is a polynomial in p with nonnegative coefficients. Define

$$\beta_\lambda(S;p) = \sum_{T\subseteq S} (-1)^{|S-T|}\alpha_\lambda(T;p).$$

Since the lattice of subgroups of a finite abelian group is modular (see, e.g., [**1**, p. 42]), it follows, e.g. from [**4**, 2.2], that for fixed p we have $\beta_\lambda(S;p) \ge 0$. We claim in fact that $\beta_\lambda(S;p)$ has nonnegative coefficients as a polynomial in p. The following proof is due to R. Stanley and inspired the proof of the main theorem of this paper.

First notice that (2) and (3) yield

$$h_{a_1}(x)h_{a_2-a_1}(x)\cdots h_{n-a_j}(x) = \sum_{\lambda\vdash n} \alpha_\lambda(S;t^{-1})t^{n(\lambda)}P_\lambda(x;t). \tag{4}$$

Now consider the skew shape [**10**, p. 4] ρ having a_1 squares in row one, $a_2 - a_1$ squares in row two, ..., $n-a_j$ squares in row $j+1$, with consecutive rows overlapping by one square. (Thus ρ is a border strip [**10**, p. 31].) Expand the skew Schur function s_ρ [**10**, p. 39] in terms of the homogeneous symmetric functions using the Jacobi-Trudi identity [**10**, p. 40]. Then, using (4), rewrite s_ρ as a linear combination of Hall-Littlewood symmetric functions to obtain

$$s_\rho(x) = \sum_{\lambda\vdash n} \beta_\lambda(S;t^{-1})t^{n(\lambda)}P_\lambda(x;t).$$

The result of Lascoux and Schützenberger, together with the fact that s_ρ is a nonnegative linear combination of Schur functions (see, e.g., [**10**, p. 68]), now is seen to imply that $\beta_\lambda(S;p)$ has nonnegative coefficients.

In [**5**, Chapter 2] we provide a simple combinatorial proof which uses only the expression given in (1) for the number of subgroups of type ν in a finite abelian p-group of type λ.

REMARK. Lascoux and Schützenberger define a statistic, charge, on tableaux T of shape ρ and weight λ such that

$$K_{\rho,\lambda}(t) = \sum_T t^{\mathrm{charge}(T)}.$$

The definition of charge as well as its relationship to Kostka polynomials $K_{\rho,\lambda}(t)$ extend naturally when ρ is a skew shape. (See [**5**, Chapter 3].) Hence, injections $\psi_{\lambda,k}$ from border strip tableaux of weight $\lambda \vdash n \geq 2k$ with $k-1$ squares in row one and $n-k+1$ squares in row two to border strip tableaux with k squares in row one and $n-k$ squares in row two, such that charge($\psi(T)$)=charge(T), provide a combinatorial version of the proof of our main theorem. We describe such an injection ψ: Add an entry ∞ at the end of row one of T. Slide row one of T to the left above row two until some entry y is immediately above an entry x with $x \leq y$. Stop sliding; select the leftmost such pair of entries x and y. The entries in row one of $\psi(T)$ are the entries in row one of T together with the entry x from row two of T.

This injection is inspired by Schützenberger's Jeu de Taquin (see, e.g. [**5**, Chapter 3]).

5. Open problems. An immediate consequence of our main result is that the lattice of subgroups (ordered by inclusion) of any finite abelian group is rank-unimodal. (The product of two rank-unimodal, rank-symmetric, finite, graded posets is again rank-unimodal and rank-symmetric. See, e.g., [**7**].) The lattice of subgroups of a finite abelian p-group of type $\lambda = (1, 1, \ldots, 1)$ is a symmetric chain order (see, e.g., [**1**, 8.63, p. 433]. It has been conjectured that this result holds when $\lambda = (r, r, \ldots, r)$. A symmetric chain order is not only rank-unimodal and rank-symmetric, but also has property S, a strong version of the Sperner property [**7**]. In general (e.g. for $\lambda = (2, 1)$), the lattice of subgroups of a finite abelian p-group need not even have the Sperner property. For further information concerning the relationship between rank-unimodality and rank-symmetry, Sperner-type properties, and chain decomposition properties, see [**7**].

One can ask whether the lattice of subgroups of a finite abelian p-group is rank-log-concave. (This is easily seen to be true for type $\lambda = (1, 1, \ldots, 1) \vdash n$ when $\alpha_\lambda(k; p)$ is the p-binomial coefficient $\left[{n \atop k}\right]_p$.) In fact, calculations invite the stronger conjecture that

$$(\alpha_\lambda(k; p))^2 - \alpha_\lambda(k-1; p)\alpha_\lambda(k+1; p)$$

has nonnegative coefficients as a polynomial in p, for all $\lambda \vdash n > k > 0$. This conjecture is not even settled for p-binomial coefficients, but we have the weaker result that

$$\left(\begin{bmatrix} n \\ k \end{bmatrix}_p\right)^2 - p \begin{bmatrix} n \\ k-1 \end{bmatrix}_p \begin{bmatrix} n \\ k+1 \end{bmatrix}_p \tag{5}$$

has nonnegative coefficients. Our argument relies on a combinatorial expression, due to Carlitz [**6**], for the p-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_p = \sum_{w \in S(1^k 2^{n-k})} p^{\mathrm{inv}(w)}$$

where $S(1^k 2^{n-k})$ is the set of permutations of the multiset $\{1^k, 2^{n-k}\}$ and $\mathrm{inv}(w)$ is the inversion number of the multiset permutation $w = w_1 w_2 \cdots w_n$. We construct an injection

$$\begin{array}{rccc} \varphi : & S(1^{k-1}2^{n-k+1}) \times S(1^{k+1}2^{n-k-1}) & \rightarrow & S(1^k 2^{n-k}) \times S(1^k 2^{n-k}) \\ & (\pi, \sigma) & \mapsto & (\pi_L \sigma_R,\ \sigma_L \pi_R) \end{array}$$

using an idea of Bhatt and Leiserson [**3**]. Just define

$$\varphi(\pi,\sigma) = (\pi_1 \cdots \pi_i \sigma_{i+1} \cdots \sigma_n,\ \sigma_1 \cdots \sigma_i \pi_{i+1} \cdots \pi_n)$$

where i is smallest with $\pi_1 \cdots \pi_i \sigma_{i+1} \cdots \sigma_n \in S(1^k 2^{n-k})$. Observe that

$$\operatorname{inv}(\pi_L \sigma_R) + \operatorname{inv}(\sigma_L \pi_R) = \operatorname{inv}(\pi) + \operatorname{inv}(\sigma) + 1.$$

For a detailed proof and a straightforward generalization see [**5**]. Sagan [**11**] independently discovered the idea behind the injection φ and gives injective proofs of log-concavity (in k) of the Stirling numbers, $c(n,k)$ and $S(n,k)$, as well as the binomial coefficients $\binom{n}{k}$. Our result in (5) is weaker than the conjecture that $([{n \atop k}]_p)^2 - [{n \atop k-1}]_p[{n \atop k+1}]_p$ has nonnegative coefficients; it is weaker because the sequence of coefficients in $[{n \atop l}]_p$ (hence in $([{n \atop k}]_p)^2$ and in $[{n \atop k-1}]_p[{n \atop k+1}]_p$) is symmetric and unimodal, and the degree of $([{n \atop k}]_p)^2$ exceeds the degree of $[{n \atop k-1}]_p[{n \atop k+1}]_p$ by 2. In [**13**], Stanley gives an elegant proof that $[{n \atop k}]_p$ has unimodal coefficients; it is not known whether $\alpha_\lambda(k;p)$ has unimodal coefficients for all λ and k. See [**5**] for tables of these polynomials.

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