ON LARGE ZSIGMONDY PRIMES

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ABSTRACT. If a and m are integers greater than 1, then a large Zsigmondy prime is a prime \( l \) such that \( l \mid a^m - 1 \), \( l \nmid a^i - 1 \) for \( 1 < i < m - 1 \) and either \( l^2 \nmid a^m - 1 \) or \( l > m + 1 \). The main result of this paper lists all the pairs \((a, m)\) for which no large Zsigmondy prime exists.

1. Introduction. Let \( a \) and \( m \) be integers greater than 1. A Zsigmondy prime for \((a, m)\) is a prime \( l \) such that \( l \mid a^m - 1 \) but \( l \nmid a^i - 1 \) for \( 1 < i < m - 1 \). A well-known theorem of Zsigmondy asserts that Zsigmondy primes exist except if \((a, m) = (2, 6)\) or \( m = 2 \) and \( a = 2^k - 1 \) (see [5]). This paper contains a refinement of this result.

Observe that if \( l \) is a Zsigmondy prime for \((a, m)\), then \( a \) has order \( m \) modulo \( l \) and so \( l \equiv 1 \pmod{m} \). Thus \( l \geq m + 1 \). A prime \( l \) is a large Zsigmondy prime for \((a, m)\) if \( l \) is a Zsigmondy prime for \((a, m)\) and either \( l > m + 1 \) or \( l^2 \nmid a^m - 1 \).

**Theorem A.** If \( a \) and \( m \) are integers greater than 1, then there exists a large Zsigmondy prime for \((a, m)\) except in the following cases.

(i) \( m = 2 \) and \( a = 2^s3^t - 1 \) for some natural number \( s \), and \( t = 0 \) or 1.
(ii) \( a = 2 \) and \( m = 4, 6, 10, 12 \) or 18.
(iii) \( a = 3 \) and \( m = 4 \) or 6.
(iv) \((a, m) = (5, 6)\).

If \( n \) is a natural number and \( l \) is a prime, let \( |n|_l \) denote the \( l \)-part of \( n \). In other words \( |n|_l = l^k \) where \( l^k \mid n \) but \( l^{k+1} \nmid n \). Thus a Zsigmondy prime \( l \) for \((a, m)\) is large if and only if \( |a^m - 1|_l > m + 1 \).

An argument similar to that used to prove Theorem A also yields the following result.

**Theorem B.** Let \( N \) be a positive integer. Then for all but a finite number of pairs of integers \((a, m)\) with \( a > 1 \) and \( m > 2 \), there exists a Zsigmondy prime \( l \) with \( |a^m - 1|_l > mN + 1 \).

The proof of Theorem B actually yields a bound which can be used to find the exceptions in Theorem B. This method is used to prove Theorem A.

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Zsigmondy's Theorem was independently, but later, discovered by Birkhoff and Vandiver [2]. Of course it follows from Theorem A by checking that in cases (ii), (iii), and (iv) a Zsigmondy prime exists except when \((a, m) = (2, 6)\).

Artin gave an elegant proof of the original result in [1]. The proof of Theorem A given here uses his approach and is based on his work.

The special case of Theorem A for \(a \geq 3\) was proved in [3]. However there is no appreciable simplification in handling the general case rather than the case \(a = 2\), so the proof given here is independent of earlier work and is self-contained.

The motivation for Theorem A comes from the theory of finite groups and is needed for some forthcoming work [4]. The relevance of Theorem A and Zsigmondy's Theorem for the theory of finite groups can also be seen for instance in [1, 3].

I am indebted to the referee for suggesting Lemma 2.1. This is stronger than my earlier result and shortens some of the arguments below.

### 2. Some preliminary results

For any natural number \(m\) let \(\Phi_m(x)\) denote the \(m\)th cyclotomic polynomial and let \(P(m)\) denote the largest prime which divides \(m\). Let \(\varphi(m) = \text{degree of } \Phi_m(x)\) denote the Euler function.

**Lemma 2.1.** Let \(a, m\) be integers with \(a > 1\) and \(m > 2\). Let \(S\) be the set of all Zsigmondy primes for \((a, m)\). Let \(Z = \prod_{l \in S} |\Phi_m(a)|_l\). Then
\[
(2.1) \quad \Phi_m(a) | ZP(m).
\]

**Proof.** This is a direct consequence of [1, Lemma 1]. \(\Box\)

**Lemma 2.2.** Let \(a, m\) be integers with \(a > 1\) and \(m > 2\). Let \(N\) be a positive integer. Suppose that if \(l\) is a Zsigmondy prime for \((a, m)\) then \(|a^m - 1|_l \leq Nm + 1\). Then
\[
(2.2) \quad |\Phi_m(a)| \leq (Nm + 1)^NP(m).
\]

**Proof.** There are at most \(N\) Zsigmondy primes \(l \leq Nm + 1\). Thus if \(Z\) is defined as in Lemma 2.1 then \(Z \leq (Nm + 1)^N\). The result follows from Lemma 2.1. \(\Box\)

**Lemma 2.3.** Let \(a, m\) be integers with \(a > 1\) and \(m > 2\). Suppose there exists no large Zsigmondy prime for \((a, m)\). Then
\[
(2.3) \quad |\Phi_m(a)| \leq (m + 1)P(m).
\]

**Proof.** Let \(N = 1\) in Lemma 2.2. \(\Box\)

The proofs of Theorems A and B are based on the fact that (2.2) and (2.3) only hold for a finite number of cases.

For any positive real number \(r\) define
\[
L_m(r) = \min_{|u| = r} |\Phi_m(u)|.
\]

Here \(u\) ranges over all complex numbers with \(|u| = r\). Since this set is compact, the minimum exists.

**Lemma 2.4.** Let \(m = nt\), where \(n\) is relatively prime to \(t\). Let \(r\) be a positive real number. Then \(L_m(r) \geq (L_n(r))^{\varphi(t)}\).

**Proof.** Let \(u\) be a complex number with \(|u| = r\). Then
\[
|\Phi_m(u)| = \left| \prod \Phi_n(ue) \right|
\]
where \( \epsilon \) ranges over all the primitive \( t \)th roots of \( 1 \). As \( |ue| = |u| \), this implies that
\[
|\Phi_m(u)| \geq |L_n(r)|^{\varphi(t)}.
\]
The result follows as \( u \) was arbitrary with \( |u| = r \).

The next two elementary results will be needed later.

**Lemma 2.5.** Let \( m \) be a natural number. Then \( \varphi(m) \geq \frac{1}{2} \sqrt{m} \).

**Proof.** If \( p \) is a prime then \( \varphi(p^b) = (p - 1)p^{b-1} \) for any natural number \( b \).
Thus \( \varphi(p^b) \geq p^{b/2} \) if \( p \geq 2 \) and \( \varphi(2^b) \geq 2^{b/2-1} \). This proves the result as \( \varphi(nt) = \varphi(n)\varphi(t) \) for \( n, t \) relatively prime.

**Lemma 2.6.** Let \( 0 < n < m \). Then \( f(x) = (x^m - 1)/(x^n + 1) \) is monotone increasing for \( x > 0 \).

**Proof.** If \( x > 0 \) then
\[
f'(x) = \frac{(m-n)x^{m+n-1} + mx^{m-1} + nx^{n-1}}{(x^n + 1)^2} > 0.
\]

**3. The proof of Theorem B.**

**Lemma 3.1.** Let \( r \geq 2 \) be a real number. Let \( m, N \) be natural numbers, suppose that
\[
L_m(r) \leq (Nm + 1)^N\varphi(m).
\]
Then \( (r - 1) \leq (2N + 1)^{2(N+1)} \).

**Proof.** By Lemma 2.4
\[
(Nm + 1)^N\varphi(m) \geq L_m(r) \geq L_1(r) \varphi(m) = (r - 1)\varphi(m).
\]
If \( (r - 1) > (2N + 1)^{2(N+1)} \) this implies that
\[
(Nm + 1)^{N+1} > (Nm + 1)^N m \geq (Nm + 1)^N\varphi(m) > (2N + 1)^{2(N+1)}\varphi(m).
\]
Hence by Lemma 2.5
\[
(Nm + 1) > (2N + 1)^{2\varphi(m)} \geq (2N + 1)^{\varphi(m)}
\]
which is impossible for \( m \geq 1 \).

**Lemma 3.2.** Let \( r \geq 2 \) be a real number. Let \( m, N \) be natural numbers such that \( m > 1 \) and \( m \neq 2n \) with \( n \) odd. Suppose that
\[
L_m(r) \leq 2^N(Nm + 1)^NP(m).
\]
Then there exists \( C = C(N) \), depending only on \( N \), with \( m < C \).

**Proof.** Suppose first that \( m = p^b \) is a prime power. By assumption, \( p^b > 2 \).
There exists a complex number \( u \) with \( |u| = r \) such that
\[
4^N(Np^b + 1)^NP \geq \left| \frac{u^{p^b} - 1}{u^{p^b-1} - 1} \right| \geq \frac{r^{p^b} - 1}{r^{p^b-1} + 1} \geq \frac{1}{3} \frac{r^{p^b} - 1}{r^{p^b-1} - 1}
\]
\[
\geq \frac{1}{3} \frac{r^{p^b-p^{b-1}}} {r^{p^b-2} + 1} \geq \frac{1}{3} \frac{r^{p^b/2}} {r^{p^b/2} - 1} \geq \frac{1}{3} 2^{p^b/2}.
\]
This implies the existence of \( C_0 = C_0(N) > 10^{20} \), depending only on \( N \), such that \( p^b < C_0 \).
We will now prove the result with \( C = C_0^k \), where \( k = \pi(C_0) \) is the number of primes \( p < C_0 \).

Suppose \( m > C \). Then \( m = p^b t \) for some prime \( p \) with \( p^b \geq C_0 \) and \( p \nmid t \).

Therefore

\[
L_{p^b}(r) > 2^N(Np^b + 1)^N p.
\]

Hence Lemma 2.4 implies that

\[
2^N(Np^b t + 1)^N p \Omega(t) \geq \left\{ L_{p^b}(r) \right\}^\psi(t) > 2^Np(Np^b + 1)^\psi(t).
\]

By assumption, \( t > 2 \) and so \( \psi(t) \geq 2 \). Thus

\[
2^Npt(2Np^b t)^N > 2^{2Np^b}(Np^b)^{(Np^b)^\psi(t)}.
\]

Hence

\[
\delta(2Np^b t)^N > 2^Np(Np^b)^{(Np^b)^\psi(t)} > \left\{ 2(Np^b)^{(Np^b)^\psi(t)} \right\}^N.
\]

After taking \( N \)th roots, Lemma 2.5 implies that

\[
(3.1) \quad Np^b t^2 > (Np^b)^\psi(t) \geq (Np^b)^{y/(y^2 - 1)}.
\]

Let \( y = Np^b \). By (3.1) \( y^2 t^2 > y^{y/(y^2 - 1)} \). As \( y \geq p^b > C_0 > 10^{20} \) this implies that if \( t \geq 5 \) then \( 25y > y^{y/(y^2 - 1)} \). Hence

\[
25 > y^{y/(y^2 - 1)/2} > y^{1/10} > 100
\]

which is not the case. Since \( t \neq 2t_0 \) with \( t_0 \) odd, (3.1) implies that one of the following holds:

\[
\begin{align*}
t &= 4 & \text{and} & \quad 16y > y^2, \\
t &= 3 & \text{and} & \quad 9y > y^2.
\end{align*}
\]

None of these can hold as \( y > 10^{20} \).

**Lemma 3.3.** Let \( a \geq 2 \) be a natural number. Let \( m, N \) be natural numbers with \( m > 2 \). Suppose that

\[
|\Phi_m(a)| \leq (Nm + 1)^NP(m).
\]

Then there exists \( C = C(N) \), depending only on \( N \), with \( m < C \).

**Proof.** If \( m \neq 2n \) with \( n \) odd this follows directly from Lemma 3.2. Suppose that \( m = 2n \) with \( n \) odd. Then \( \Phi_m(u) = \Phi_n(-u) \) for any complex number \( u \) and \( P(m) = P(n) \). Thus \( L_m(a) = L_n(a) \). Hence Lemma 3.2 implies that if no such \( C = C(N) \) exists then

\[
|\Phi_m(a)| \geq L_n(a) > 2^N(Nn + 1)^N P(n)
\]

\[
> (2Nn + 1)^N P(n) = (Nm + 1)^N P(m).
\]

**Proof of Theorem B.** This is an immediate consequence of Lemmas 2.2, 3.1, and 3.3.

**4. The proof of Theorem A.** The proof of Theorem A is analogous to that of Theorem B but the relevant inequalities need to be considered more precisely.
Lemma 4.1. Suppose that \( m = nt \) with \( n > 1 \) and \( n \) relatively prime to \( t \). Assume that \( m \neq 2 \pmod{4} \). Let \( r \) be a real number such that \( L_n(r) > 2(n+1)P(n) \). Then \( L_m(r) > 2(m+1)P(m) \).

Proof. By induction on the number of primes dividing \( t \) it may be assumed that \( t = p^b \) for some prime \( p \). By assumption \( n, t \neq 2 \pmod{4} \) and so \( t > 2 \) and \( n > 2 \).

Suppose the result is false. Then by Lemma 2.4

\[
2(np^b + 1)pP(n) \geq L_m(r) \geq L_n(r)p^b-p^{b-1} > \left\{2(n + 1)P(n)\right\}p^b-p^{b-1}.
\]

Thus

\[
4np^2 > 2(np^b + 1)p > \left\{2(n + 1)\right\}p^b-p^{b-1}P(n)p^b-p^{b-1-1}
\]

\[
> \left\{2(n + 1)\right\}p^b-p^{b-1-1}2p^b-p^{b-1-1-1}.
\]

Let \( y = \frac{1}{2}p^b \). As \( p^b-p^{b-1} \geq \frac{3}{2}p^b, (4.1) \) implies that \( 16y^2 > 2^{2-1}(n + 1)^2 \). Hence

\[
16y^2 > 2^{2-1}(n + 1)^2 \geq 2^{2-1}(n + 1)^2 = 2^{2-3}.
\]

Thus \( p^b = y^2 < 7 \). Hence \( p^b = 3, 4 \) or \( 5 \) contrary to (4.1). \( \Box \)

Lemma 4.2. Let \( r > 2 \) be a real number. Let \( m \) be an integer such that \( m \neq 2n \) for \( n \) odd and \( m > 1 \). Then

\[
L_m(r) > 2(m+1)P(m)
\]

except possibly in the following cases.

1. \( m = 3 \) and \( r < 6 \).
2. \( m = 4 \) and \( r < 5 \).
3. \( m = 12 \) and \( r < 4 \).
4. \( r < 3 \) and \( m = 5, 7, 8, 9, 15 \) or \( 20 \).

Proof. Assume that \( m \) does not satisfy the conclusions of the lemma. In particular (4.2) does not hold.

Suppose first that \( m = p^b > 2 \) for some prime \( p \). Then for suitable \( u \) with \(|u| = r\), Lemma 2.6 implies that

\[
2(p^b + 1)p \geq \frac{u^{p^b} - 1}{u^{p^b-1} - 1} \geq \frac{r^{p^b} - 1}{r^{p^b-1} + 1} \geq \frac{2^{p^b} - 1}{2^{p^b-1} + 1}.
\]

Suppose that \( m = p \) is a prime. Then (4.3) implies that \( 6p(p + 1) \geq 2^p - 1 \). Hence \( p < 11 \). Thus \( p = 3, 5 \) or \( 7 \) and (4.3) implies that

\[
2p(p + 1) \geq \frac{r^p - 1}{r + 1}.
\]

If \( p = 7 \) this yields that \( 112 \geq (r^7 - 1)/(r + 1) \) and so \( r < 3 \). If \( p = 5 \) then \( 60 \geq (r^5 - 1)/(r + 1) \) and so \( r < 3 \). If \( p = 3 \) then \( 24 \geq (r^3 - 1)/(r + 1) \) and so \( r < 6 \).

Suppose that \( m = p^b \) with \( b > 1 \). Then \( r^{p^b} \geq 4 \) and so \( r^{p^b} - 1 - 1 \geq \frac{1}{2}(r^{p^b} - 1) \). Hence (4.3) implies that

\[
2(p^b + 1)p \geq \frac{1}{2}r^{p^b - 1} - 1 \geq \frac{1}{2}r^{p^b} - p^{b-1}.
\]

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Let \( y = \frac{1}{2}p^b \). Then \( y \leq p^b - p^{b-1} \). Thus (4.4) yields that
\[
2(2y + 1)y \geq 2^{y-1}.
\]
Hence \( y \leq 9 \) and so \( p^b \leq 18 \). Thus \( p^b = 4, 8, 9 \) or \( 16 \).

If \( p^b = 16 \) then (4.3) implies that \( 68 \geq (2^{16} - 1)/(2^8 + 1) = 2^8 - 1 \) which is not the case.

If \( p^b = 9 \) then (4.4) implies that \( 120 \geq r^6 \) and so \( r < 3 \).

If \( p^b = 8 \) then (4.3) implies that \( 36 \geq (r^8 - 1)/(r^4 + 1) = r^4 - 1 \) and so \( r < 3 \).

If \( p^b = 4 \) then (4.3) implies that \( 20 \geq (r^4 - 1)/(r^2 + 1) = r^2 - 1 \) and so \( r < 5 \).

In summary, we have shown that if \( m = p^b \), then \( m \) satisfies the conclusion of the lemma. In particular (4.2) holds unless \( m = 4, 8, 3, 9, 5 \) or \( 7 \). Hence Lemma 4.1 implies that
\[ m|2^3 \cdot 3^2 \cdot 5 \cdot 7. \]

Suppose that \( m = 7t \). Then \( t|2^3 \cdot 3^2 \cdot 5 \). As (4.2) does not hold, Lemma 2.4 implies that for suitable \( u \) with \( |u| = r \).

\[
14(7t + 1) \geq \left| \frac{u^7 - 1}{u - 1} \right| \frac{\varphi(t)}{2^{7 - 1}} \frac{1}{r + 1} = \left( \frac{r^7 - 1}{r + 1} \right)^{\varphi(t)}. \tag{4.5}
\]

This yields the following inequalities.

If \( t = 5 \) then \( 14 \cdot 36 \geq \left( (r^7 - 1)/(r + 1) \right)^4 \).

If \( t = 9 \) then \( 14 \cdot 64 \geq \left( (r^7 - 1)/(r + 1) \right)^6 \).

If \( t = 8 \) then \( 14 \cdot 57 \geq \left( (r^7 - 1)/(r + 1) \right)^4 \).

None of these can hold for \( r \geq 2 \). Hence 5 \( \not\equiv \mod 4 \) either \( t = 3 \) or \( 4 | t \).

Suppose \( t = 4 \); then (4.5) becomes \( 14 \cdot 29 \geq \left( (r^7 - 1)/(r + 1) \right)^2 \), which is impossible for \( r \geq 2 \). Thus \( 4 \equiv t \) by Lemma 4.1 and so \( t = 3 \). Now (4.5) becomes \( 14 \cdot 22 \geq \left( (r^7 - 1)/(r + 1) \right)^2 \), which is impossible for \( r > 2 \).

Therefore it may be assumed that
\[ m|2^3 \cdot 3^2 \cdot 5. \]

Suppose that \( m = 5t \). Then \( t|2^3 \cdot 3^2 \). As (4.2) does not hold, Lemma 4.1 implies that for suitable \( u \) with \( |u| = r \).

\[
10(5t + 1) \geq \left| \frac{u^5 - 1}{u + 1} \right| \frac{\varphi(t)}{2^{5 - 1}} \frac{1}{r + 1} = \left( \frac{r^5 - 1}{r + 1} \right)^{\varphi(t)}. \tag{4.6}
\]

This yields the following inequalities.

If \( t = 9 \) then \( 460 \geq \left( (r^5 - 1)/(r + 1) \right)^6 \).

If \( t = 8 \) then \( 410 \geq \left( (r^5 - 1)/(r + 1) \right)^4 \).

Neither of these can hold for \( r \geq 2 \). Thus Lemma 4.1 implies that \( 9 \equiv t \) and \( 8 \equiv t \). Hence \( t \equiv 2 \mod 4 \) and so \( t = 3, 4, \) or 12.

If \( t = 12 \) then (4.6) becomes \( 610 \geq \left( (r^5 - 1)/(r + 1) \right)^4 \), which is impossible for \( r \geq 2 \).

If \( t = 4 \) then (4.6) becomes \( 210 \geq \left( (r^5 - 1)/(r + 1) \right)^2 \). Hence \( r < 3 \).

If \( t = 3 \) then (4.6) becomes \( 160 \geq \left( (r^5 - 1)/(r + 1) \right)^2 \). Hence \( r < 3 \).
Therefore it may be assumed that
\[ m \mid 2^3 \cdot 3^2. \]

Suppose that \( m = 9t \) with \( t = 4 \) or 8. As (4.2) does not hold, Lemma 4.1 implies that for suitable \( u \) with \( |u| = r \)
\[
6(9t + 1) \geq \left| \frac{u^9 - 1}{u^3 - 1} \right|^{\varphi(t)} \geq \left( \frac{r^9 - 1}{r^3 + 1} \right)^{\varphi(t)}.
\]
This yields the following inequalities.

If \( t = 8 \) then \( 6 \cdot 73 \geq ((2^3 - 1)/9)^4 \).

If \( t = 4 \) then \( 6 \cdot 37 \geq ((2^3 - 1)/9)^2 \).

Neither of these holds and so \( 9 + m \). Thus \( m \mid 24 \). Hence \( m = 12 \) or 24.

If \( m = 24 \) and (4.2) does not hold, Lemma 4.1 implies that for suitable \( u \) with \( |u| = r \),
\[
150 \geq \left| \frac{u^8 - 1}{u^4 - 1} \right|^2 = |u^4 + 1|^2 \geq (r^4 - 1)^2.
\]
This is impossible for \( r \geq 2 \).

If \( m = 12 \) and (4.2) does not hold, Lemma 4.1 implies that for suitable \( u \) with \( |u| = r \),
\[
78 \geq \left| \frac{u^4 - 1}{u^2 - 1} \right|^2 = |u^2 + 1|^2 \geq (r^2 - 1)^2.
\]
Thus \( r < 4 \).

**Lemma 4.3.** Suppose that \( a, m \) are integers with \( a \geq 2 \) and \( m \geq 3 \). Then
\[
|\Phi_m(a)| > (m + 1)P(m)
\]
extcept possibly in the following cases.

(i) \( m = 3 \) or 6 and \( a < 6 \).

(ii) \( m = 4 \) and \( a < 5 \).

(iii) \( m = 12 \) and \( a < 4 \).

(iv) \( a < 3 \) and \( m = 5, 7, 8, 9, 10, 14, 15, 18, 20 \) or 30.

**Proof.** If \( m \neq 2n \) with \( n \) odd, then the result follows directly from Lemma 4.2. Suppose that \( m = 2n \) with \( n \) odd. Then \( \Phi_m(u) = \Phi_n(-u) \) for any complex number \( u \) and \( P(m) = P(n) \). Thus \( L_m(a) = L_n(a) \). Hence if (4.2) holds for \( n \) then
\[
|\Phi_m(a)| \geq L_n(a) > 2(n + 1)P(n) = (2n + 2)P(m) > (m + 1)P(m).
\]
Suppose that (4.2) does not hold for \( n \). By Lemma 4.2 one of the following cases must occur:
\[
n = 3, \quad m = 6, \quad \text{and} \quad a < 6.
\]
\[
a < 3 \quad \text{and} \quad n = 5, 7, 9 \text{ or } 15.
\]
Thus \( m = 10, 14, 18 \) or 30. \( \Box \)

**Proof of Theorem A.** Suppose that \( m = 2 \). Since 2 is the greatest common divisor of \( a - 1 \) and \( a + 1 \) it follows that any odd prime dividing \( a + 1 \) is a Zsigmondy prime. This implies the result for \( m = 2 \). Suppose now that \( m \geq 3 \).
Assume that there exists no large Zsigmondy prime for \((a, m)\). By Lemmas 2.3 and 4.3 \((a, m)\) is one of the cases listed in Lemma 4.3. We will exhibit a large Zsigmondy prime \(l\) in each case that is not listed in the statement of Theorem A. This will conclude the proof.

\[
\begin{array}{c|cccc}
  m = 3, & a & 2 & 3 & 4 & 5 \\
  l & 7 & 13 & 7 & 31 \\
\end{array}
\]

\[
\begin{array}{c|c}
  m = 6, & a & 4 \\
  l & 13 \\
\end{array}
\]

\[
\begin{array}{c|c}
  m = 4, & a & 4 \\
  l & 17 \\
\end{array}
\]

\[
\begin{array}{c|c}
  m = 12, & a & 3 \\
  l & 73 \\
\end{array}
\]

\[
\begin{array}{c|cccccccccc}
  a = 2, & m & 5 & 7 & 8 & 9 & 14 & 15 & 20 & 30 \\
  l & 31 & 127 & 17 & 73 & 43 & 151 & 41 & 331 \\
\end{array}
\]

\[
\square
\]

**References**