PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 1, January 1988

A FURTHER REFINEMENT OF THE BRUHAT DECOMPOSITION

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(Communicated by Bhama Srinivasan)

ABSTRACT. Kawanaka obtained explicit formulas for the structure constants in the Hecke algebra $H(G(q), B(q))$ of a finite Chevalley group $G(q)$. This note contains a geometric interpretation of these formulas, involving decompositions of Bruhat cells, in connected reductive algebraic groups.

Introduction. Let G be a connected reductive affine algebraic group over an algebraically closed field k, defined and split over a perfect subfield $k_0 \subseteq k$. Let $W = N_G(T)/T$ be the Weyl group of G with respect to a fixed maximal split torus T, contained in a Borel subgroup B, defined and split over k_0. Let (w, w', w'') denote an arbitrary triple of elements of W, and let $\dot{w}, \dot{w}', \dot{w}''$ denote their representatives in $N_G(T)(k_0)$. Let $U = R_u(B)$, and let $U_x = U \cap {}^xB_-$ for $x \in W$, where B_- is the Borel subgroup opposed to B. The main result in this note is an explicit decomposition of the set

$$U(w, w', w'') = \{u \in U_w : u\dot{w}B \cap \dot{w}''U_{(w')^{-1}}(\dot{w}')^{-1} \neq \varnothing\}$$

as a disjoint union of subsets $\{U_\tau\}$ indexed by certain subexpressions τ of a reduced expression of w as a product of the distinguished generators $\{s_1, \ldots, s_n\}$ of W. Each subset U_τ is a locally closed k_0-subvariety of U_w and is isomorphic, as a variety defined over k_0, to a product $G_a^{a(\tau)} \times G_m^{b(\tau)}$, where G_a and G_m denote the additive and multiplicative groups of k, respectively, and $a(\tau)$ and $b(\tau)$ are nonnegative integers defined by the subexpression τ.

In case $w' = w_0$, the element of W of maximal length, a similar problem was solved by Deodhar [**3**]. In the general case, the result gives a geometric interpretation of a result of Kawanaka [**4**]. He considered the situation where k_0 is a finite field, and proved that the structure constant $[b_w b_{w'} : b_{w''}]$ for standard basis elements of the Iwahori algebra (or Hecke algebra) $H(G(k_0), B(k_0))$ is given by

$$[b_w b_{w'} : b_{w''}] = \sum_\tau q^{a(\tau)}(q-1)^{b(\tau)}.$$

A corollary of our result gives a description of each set $U_\tau(k_0)$ in terms of root subgroups of $G(k_0)$, which implies Kawanaka's formula, and will be applied in a subsequent paper to describe the multiplication in the Hecke algebra of the Gelfand-Graev representation of $G(k_0)$, in case k_0 is a finite field, and the center of G is

Received by the editors September 2, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20G15.
Key words and phrases. Reductive algebraic groups, Bruhat decomposition, Hecke algebra.
The author's research was supported in part by NSF grant DMS-8318037-01.

connected. Another interesting problem would be to describe the closure patterns of the subsets $\{U_\tau\}$.

The author thanks N. Kawanaka and V. Deodhar for discussions, at various times, on these ideas. The idea of seeking such a decomposition of the sets $U(w, w', w'')$ comes from the author's attempt to understand the structure of the Hecke algebra of the Gelfand-Graev representation of $G(k_0)$ for a finite field k_0 (see Yokonuma [**6**], Steinberg [**5**], and Chang [**2**]).

1. Statements of results. Besides the notation given in the Introduction, we require some other basic facts (for proofs, see Borel and Tits [**1**]). Let Φ denote the root system of G with respect to T, $\{U_\alpha : \alpha \in \Phi\}$ the root subgroups, and $\Phi^\pm$ the sets of positive and negative roots defined by the Borel subgroups B and B^-. $\Delta = \{\alpha_i : 1 \le i \le n\}$ denotes the set of simple roots in Φ^+, corresponding to the set of fundamental reflections $S = \{s_i : 1 \le i \le n\}$. For each $x \in W$, let $\psi_x = \Phi^+ \cap x\Phi^-$; then $|\psi_x| = l(x)$, where $l(x)$ is the length of x in the Coxeter system (W, S). We have

$$U_x = \prod_{\alpha \in \psi_x} U_\alpha \tag{1.1}$$

for an arbitrary ordering of the roots in ψ_x, and, in any order, the map of the direct product of the $\{U_\alpha, \alpha \in \psi_x\}$ to U_x is an isomorphism of k_0-varieties. Moreover,

$$U = U_x U_{xw_0} = U_{xw_0} U_x, \tag{1.2}$$

and the map from the direct product $U_x \times U_{xw_0}$ to U in an isomorphism of k_0-varieties.

We have

$$G = \bigcup_{x \in W} U_x \dot{x} B \tag{1.3}$$

for a fixed cross section $\{\dot{x} : x \in W\}$ in $N_G(T)(k_0)$, and for each x there exists an isomorphism of k_0-varieties $U_x \times B \cong U_x \dot{x} B$ given by the product map. A decomposition similar to (1.3) holds for the group of rational points $G(k_0)$.

Now let (w, w', w'') denote an arbitrary triple of elements of W, which will remain fixed, and let $s_l \cdots s_1$ be a fixed reduced expression of w with factors $s_i \in S$. We define

$$U(w, w', w'') = \{u \in U_w : u\dot{w}B \cap \dot{w}'' U_{(w')^{-1}} (\dot{w}')^{-1} \neq \varnothing\}. \tag{1.4}$$

As in Deodhar [**3**], a *subexpression* τ of the reduced expression $s_l \cdots s_1$ is a sequence $\tau = (\tau_l, \ldots, \tau_1, \tau_0)$ of elements of W with the property that $\tau_i \tau_{i-1}^{-1} \in \{1, s_i\}$ for $1 \le i \le l$ and $\tau_0 = 1$. Thus the set of terminal elements $\{\tau_l\}$ of subexpressions of $s_l \cdots s_1$ coincides with the set of elements $x \in W$ such that $x \le w$ in the Bruhat order (see Deodhar [**3**]). A subexpression $\tau = (\tau_l, \ldots, \tau_1, \tau_0)$ is called a *K-sequence* relative to the triple $(s_l \cdots s_1 = w, w'w'')$ if it satisfies the conditions (2.10)(a–c) of Kawanaka [**4**]. Let $J_\tau = \{j : \tau_j \tau_{j-1}^{-1} = s_j\} \cup \{0\}$. Then τ is such a K-sequence if and only if

$$\tau_l w' = w'', \qquad l(s_p \tau_j w') < l(\tau_j w') \tag{1.5}$$

for each $j \in J_\tau$ and p in the interval between j and the next element in J_τ (or simply all $p > j$ if j is the maximal element of J_τ). For each K-sequence τ, set

$$J_\tau^- = \{j \in J_\tau : l(s_j \tau_{j'} w') < l(\tau_{j'} w')\},$$

where $j' \in J_\tau$ is the predecessor of j, and define

$$a(\tau) = |J_\tau^-|, \qquad b(\tau) = l - |J_\tau| + 1 = \operatorname{card}\{j > 0 : \tau_j \tau_{j-1}^{-1} = 1\}.$$

The main results can be stated as follows.

(1.6) THEOREM. *Let (w, w', w'') be a triple of elements of W, $s_l \cdots s_1$ a fixed reduced expression of w, and $K = K(s_l \cdots s_1, w', w'')$ the set of K-sequences relative to $(s_l \cdots s_1, w', w'')$. Then $U(w, w', w'')$ is independent of the choice of the cross section $\{\dot{x}\}$, and we have*

$$U(w, w', w'') = \bigcup_{\tau \in K} U_\tau \quad (\textit{disjoint union}),$$

where, for each τ, U_τ is a locally closed k_0-subvariety of U_w. Moreover, there exist subsets Φ_τ and Φ_τ^* of Φ^+ of cardinalities $a(\tau)$ and $b(\tau)$ respectively, such that, for each τ, there exists an isomorphism of k_0-varieties

$$U_\tau \cong \prod_{\alpha \in \Phi_\tau} U_\alpha \times \prod_{\beta \in \Phi_\tau^*} U_\beta^*$$

(where U_β^* denotes the open set $U_\beta - \{1\}$ for $\beta \in \Phi$).

In the statement of the next result, $U(w, w', w'')(k_0)$ and $U_\tau(k_0)$ denote the sets of k_0-rational points on $U(w, w', w'')$ and U_τ, respectively, for $\tau \in K$.

(1.7) COROLLARY. *Keep the notation of Theorem 1.6. Then*

$$U(w, w', w'')(k_0) = \bigcup_{\tau \in K} U_\tau(k_0) \quad (\textit{disjoint union}),$$

where for each τ, there exists a bijection

$$U_\tau(k_0) \rightleftharpoons \prod_{\alpha \in \Phi_\tau} U_\alpha(k_0) \times \prod_{\beta \in \Phi_\tau^*} U_\beta^*(k_0).$$

2. Proofs. For each simple root $\alpha_i \in \Delta$, let G_i denote the k_0-subgroup of semisimple rank one generated by $\{T, U_{\pm\alpha_i}\}$. We require the following structure equations concerning the multiplication in G_i:

(2.1) LEMMA. *For each i, $1 \le i \le n$, there exist automorphisms of k_0-varieties $f_i, g_i : U_{\alpha_i}^* \to U_{\alpha_i}^*$ such that, for each $u \in U_{\alpha_i}^*$,*

$$\dot{s}_i u \dot{s}_i = f_i(u) t_i(u) \dot{s}_i g_i(u),$$

where $t_i(u) \in T$.

The proof can readily be adapted from [**3**, Lemma 2.1].

We now begin the proof of Theorem 1.6, using induction on $l(w)$. It is clear, in all cases, that $U(w, w', w'')$ is independent of the choice of the cross section $\{\dot{x}\}$. In case $l(w) = 1$, with $w = s_i \in S$, we have

$$U(s_i, w', w'') = \begin{cases} 1 & \text{if } s_i w' = w'',\ l(s_i w') > l(w'), \\ U_{\alpha_i} & \text{if } s_i w' = w'',\ l(s_i w') < l(w'), \\ U_{\alpha_i}^* & \text{if } w' = w'',\ l(s_i w') < l(w'), \end{cases}$$

and is empty in all other cases. Using this information, Theorem 1.6 is easily verified in this case.

For the rest of the proof, $s_l \cdots s_1$ denotes a fixed reduced expression for w, with $l > 1$. We assume the result holds for all triples $(s_{l-1} \cdots s_1, x, y)$, with $x, y \in W$, and note that $U(w, w', w'')$ is contained in the k_0-subgroup U_w, which can be factored

$$U_w = U_{\alpha_l}{}^{\dot{s}_l} U_{s_l w} \tag{2.2}$$

with uniqueness, so that there exists an isomorphism of k_0-varieties $U_w \cong U_{\alpha_l} \times U_{s_l w}$.

For a given triple $(s_l \cdots s_1, w', w'')$, we consider two cases.

Case 1. $l(s_l w'') < l(w'')$. In this case, by [**4**, proof of 2.14], the set of K-sequences $K = K(s_l \cdots s_1, w', w'')$ is partitioned, $K = K_1 \cup K_2$, where K_1 consists of all sequences $\sigma = (\sigma_l, \ldots, \sigma_1, \sigma_0)$ with $\sigma' = (\sigma_{l-1}, \ldots, \sigma_1, \sigma_0) \in K(s_{l-1} \cdots s_1, w', s_l w'')$ and $\sigma_l \sigma_{l-1}^{-1} = s_l$, while K_2 consists of all sequences $\tau = (\tau_l, \ldots, \tau_1, \tau_0)$ with $\tau' = (\tau_{l-1}, \ldots, \tau_1, \tau_0) \in K(s_{l-1} \cdots s_1, w', w'')$ and $\tau_l \tau_{l-1}^{-1} = 1$.

For each sequence $\sigma \subset K_1$, we may assume the existence of $(U_{\sigma'}, \Phi_{\sigma'}, \Phi^*_{\sigma'})$ by the induction hypothesis, where $\sigma' \in K(s_{l-1} \cdots s_1, w', s_l w'')$ is the part of σ defined above. We then define

$$U_\sigma = \dot{s}_l^{-1} U_{\sigma'} \dot{s}_l, \qquad \Phi_\sigma = s_l \Phi_{\sigma'}, \qquad \Phi^*_\sigma = s_l \Phi^*_{\sigma'},$$

so, in particular, U_σ is a locally closed k_0-subvariety of U_w. We shall prove that the union X_1 of all subsets $\{U_\sigma : \sigma \in K_1\}$ is the part of $U(w, w', w'')$ contained in ${}^{s_l}U_{s_l w}$ (see (2.2)). This follows since, as is easily checked, an element $u \in U_{s_l w}$ belongs to $U(s_l w, w', s_l w'')$ if and only if $\dot{s}_l^{-1} u \dot{s}_l \in U(w, w', w'')$. The part of the induction hypothesis giving the structure of each subset $U_{\sigma'}$ in terms of the root subgroup $\{U_\alpha : \alpha \in \Phi_{\sigma'}\}$ and $\{U_\beta : \beta \in \Phi^*_{\sigma'}\}$ implies the corresponding result concerning the structure of U_σ in terms of the root subgroups from roots in Φ_σ and Φ^*_σ. This is clear since U_σ is obtained from $U_{\sigma'}$, and the root subgroups $\{U_\alpha : \alpha \in \Phi_\sigma\}$ and $\{U_\beta : \beta \in \Phi^*_\sigma\}$ are all obtained from those giving the structure of $U_{\sigma'}$, by applying the inner automorphism by $\dot{s}_l$.

We now prove that the remaining part of $U(w, w', w'')$, $X_2 = U(w, w', w'') - X_1$, is the union of subsets associated with sequences $\tau \in K_2$. We first show that for an element $u_w \in U_w$ of the form $u_w = u \dot{s}_l u_1 \dot{s}_l^{-1}$, with $u \in U^*_{\alpha_l}$, $u_1 \in U_{s_l w}$ as in (2.2), we have

$$u_w \in U(w, w', w'') \quad \text{if and only if} \quad \pi(g_l(u) u_1) \in U(s_l w, w', w'') \tag{2.3}$$

where $\pi : U \to U_{s_l w}$ is the morphism of k_0-varieties defined by the factorization $U = U_{s_l w} U_{s_l w w_0}$ given by (1.2), and $g_l : U^*_{\alpha_l} \to U^*_{\alpha_l}$ is the automorphism of k_0-varieties defined by Lemma 2.1. To begin with, we have $u_w \in U(w, w', w'')$ if and only if

$$\dot{s}_l^{-1} \dot{s}_l u \dot{s}_l u_1 \dot{s}_l^{-1} \dot{w} B \cap \dot{w}'' U_{(w')^{-1}} (\dot{w}')^{-1} \neq \varnothing.$$

Using Lemma 2.1, this is equivalent to the statement

$$\dot{s}_l^{-1} f_l(u) \dot{s}_l t g_l(u) u_1 \dot{s}_l^{-1} \dot{w} B \cap \dot{w}'' U_{(w')^{-1}} (\dot{w}')^{-1} \neq \varnothing \tag{2.4}$$

with $t = \dot{s}_l^{-1} t_l(u) \dot{s}_l \in T$ and $\dot{s}_l^{-1} f_l(u) \dot{s}_l \in U_{-\alpha_l}$.

In case $w'(w'')^{-1} s_i \alpha_i \in \Phi^-$, we see that (2.4) holds if and only if

$$g_l(u) u_1 (s_l w)^{\cdot} B \cap \dot{w}'' U_{(w')^{-1}} (\dot{w}')^{-1} \neq \varnothing \tag{2.5}$$

since $w''^{-1}U_{-\alpha_l}w'' \leq U$ by the hypothesis of Case 1, so the left factor of (2.4) can be absorbed in $U_{(w')^{-1}}$, while the element t can be conjugated through the right-hand side, and then absorbed by B. Appying the factorization (1.2) for $s_l w$ to the element $g_l(u)u_1$, we obtain (2.3) in this case.

Now let $w'(w'')^{-1}s_l\alpha_l \in \Phi^+$. Then (2.4) is equivalent to

$$tg_l(u)u_1\dot{s}_l^{-1}\dot{w}B \cap \dot{w}''u^*U_{(w')^{-1}}(\dot{w}')^{-1} \neq \varnothing \tag{2.6}$$

where

$$u^* = \dot{w}''^{-1}\dot{s}_l^{-1}f_l(u)^{-1}\dot{s}_l\dot{w}'' \in U_{(w')^{-1}w_0}.$$

Since $U_{(w')^{-1}w_0}U_{(w')^{-1}} = U_{(w')^{-1}}U_{(w')^{-1}w_0}$, and $(w')^{-1}U_{(w')^{-1}w_0}w' \leq B$, we find that (2.6) is equivalent to (2.5), completing the proof of (2.3).

After some preliminary remarks, we shall apply (2.3) to define the subsets $\{U_\tau : \tau \in K_2\}$. Since k_0 is a perfect field and ${}^{\dot{s}_l}U_{s_l w}$ is a closed k_0-subgroup of U_w, its complement Z_w is an open subvariety of U_w defined over k_0. The map $(u, u_1) \to u\dot{s}_l u_1 \dot{s}_l^{-1}$ defines an isomorphism of k_0-varieties $U^*_{\alpha_l} \times U_{s_l w} \cong Z_w$. The map $\phi\colon U^*_{\alpha_l} \times U_{s_l w} \to U_{s_l w}$ given by $\phi(u, u_1) = \pi(uu_1)$ for $u \in U^*_{\alpha_l}$ and $u_1 \in U_{s_l w}$ is clearly a morphism of varieties over k_0. Now let $v \in U_{s_l w}$. For each fixed element $u \in U^*_{\alpha_l}$, it is easily verified using the factorization (1.2) that $u_1 = \pi(u^{-1}v)$ is the unique element of $U_{s_l w}$ with the property that $\pi(uu_1) = v$. Combining these remarks, we see that the map $us_l u_1 s_l^{-1} \to \pi(g_l(u)u_1)$ is a surjective morphism of k_0-varieties from $Z_w \to U_{s_l w}$.

For each K-sequence $\tau \in K_2$, with $\tau = (\tau_l, \ldots, \tau_0)$, and $\tau' = (\tau_{l-1}, \ldots, \tau_0)$ a K-sequence for the triple $(s_{l-1} \cdots s_1, w', w'')$, we define

$$U_\tau = \{u\dot{s}_l u_1 \dot{s}_l^{-1} \in U^*_{\alpha_l}\, {}^{s_l}U_{s_l w} : \pi(g_l(u)u_1) \in U_{\tau'}\} \tag{2.7}$$

and

$$\Phi_\tau = s_l\Phi_{\tau'}, \qquad \Phi^*_\tau = s_l\Phi^*_{\tau'} \cup \{\alpha_l\}.$$

By (2.3) and the induction hypothesis, it follows that $X_2 = U(w, w', w'') - X_1$ is the disjoint union of the subsets $\{U_\tau\}$, $\tau \in K_2$.

Each subset U_τ is clearly a locally closed k_0-subvariety of Z_w, by the last statement in the paragraph preceding (2.7) and the induction hypothesis applied to $U_{\tau'}$. Now let

$$Y_\tau = \{(u, u_1) \in U^*_{\alpha_l} \times U_{s_l w} : \pi(uu_1) \in U_{\tau'}\}.$$

Then Y_τ is a locally closed subvariety of $U^*_\alpha \times U_{s_l w}$ defined over k_0, and the map $u\dot{s}_l u_1 \dot{s}_l^{-1} \to (g_l(u), u_1)$ defines an isomorphism of k_0-varieties $U_\tau \cong Y_\tau$. We next observe that the map $\xi\colon Y_\tau \to U^*_{\alpha_l} \times U_{\tau'}$ defined by $\xi(u, u_1) = (u, \pi(uu_1))$ is a surjective morphism of varieties over k_0. On the other hand, the preceding discussion shows that the map $\eta\colon U^*_{\alpha_l} \times U_{\tau'} \to Y_\tau$ given by $\eta(u, v) = (u, \pi(u^{-1}v))$ is a surjective morphism of k_0-varieties, and that ξ and η are inverses of each other. It follows that $Y_\tau \cong U^*_{\alpha_l} \times U_{\tau'}$ as k_0-varieties, and hence the isomorphism $U_\tau \cong Y_\tau$ defines an isomorphism of k_0-varieties

$$U_\tau \cong \prod_{\alpha \in \Phi_\tau} U_\alpha \times \prod_{\beta \in \phi^*_\tau} U^*_\beta,$$

using the definitions of Φ_τ and Φ^*_τ after (2.7). This completes the proof of the inductive step in Case 1.

Case 2. $l(s_l w'') > l(w'')$. In this situation, each K-sequence $\tau = (\tau_l, \ldots, \tau_0)$ relative to $(s_l \cdots s_1, w', w'')$ has the property that $\tau_l \tau_{l-1}^{-1} = s_l$, and $\tau' = (\tau_{l-1}, \ldots, \tau_0)$ is a K-sequence relative to $(s_{l-1} \cdots s_1, w', s_l w'')$. By [**4**, loc.cit.], this yields a bijection between the two sets of K-sequences. We shall prove that

$$U(w, w', w'') = U_{\alpha_l} \dot{s}_l U(s_l w, w', s_l w'') \dot{s}_l^{-1}. \tag{2.8}$$

Let $x \in U_w$, and write $x = u\dot{s}_l u_1 \dot{s}_l^{-1}$, with $u \in U_{\alpha_l}$, $u_1 \in U_{s_l w}$. Then $x \in U(w, w', w'')$ if and only if

$$u\dot{s}_l u_1 \dot{s}_l^{-1} \dot{w} B \cap \dot{w}'' U_{(w')^{-1}} (\dot{w}')^{-1} \neq \varnothing.$$

Since $l(s_l w'') > l(w'')$, we have $\dot{w}''^{-1} u \dot{w}'' \in U$, and belongs either to $U_{(w')^{-1}}$ or to $U_{(w')^{-1} w_0}$. In either case, it is easily verified, using (1.2) if necessary, that $x \in U(w, w', w'')$ if and only if $u_1 \in U(s_l w, w', s_l w'')$ which proves (2.8). Using (2.8) and the induction hypothesis, it is clear how to define the subsets $\{U_\tau\}$ for each K-sequence $\tau \in K(s_l \cdots s_1, w', w'')$, and corresponding sets of roots Φ_τ and Φ_τ^*, so that the assertions of the theorem hold. This completes the proof of Theorem 1.6.

The proof of Corollary 1.7 is obtained by following the steps of the proof of Theorem 1.6 for the group of rational points $G(k_0)$.

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