A FURTHER REFINEMENT
OF THE BRUHAT DECOMPOSITION
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ABSTRACT. Kawanaka obtained explicit formulas for the structure constants in the Hecke algebra $H(G(q), B(q))$ of a finite Chevalley group $G(q)$. This note contains a geometric interpretation of these formulas, involving decompositions of Bruhat cells, in connected reductive algebraic groups.

Introduction. Let $G$ be a connected reductive affine algebraic group over an algebraically closed field $k$, defined and split over a perfect subfield $k_0 \subseteq k$. Let $W = N_G(T)/T$ be the Weyl group of $G$ with respect to a fixed maximal split torus $T$, contained in a Borel subgroup $B$, defined and split over $k_0$. Let $(w, w', w'')$ denote an arbitrary triple of elements of $W$, and let $\check{w}, \check{w}', \check{w}''$ denote their representatives in $N_G(T)(k_0)$. Let $U = R_u(B)$, and let $U_x = U \cap xB_-$ for $x \in W$, where $B_-$ is the Borel subgroup opposed to $B$. The main result in this note is an explicit decomposition of the set

$$U(w, w', w'') = \{u \in U_w : uwB \cap \check{w}'U_{(w')}^{-1}(\check{w}')^{-1} \neq \emptyset\}$$

as a disjoint union of subsets $\{U_\tau\}$ indexed by certain subexpressions $\tau$ of a reduced expression of $w$ as a product of the distinguished generators $\{s_1, \ldots, s_n\}$ of $W$. Each subset $U_\tau$ is a locally closed $k_0$-subvariety of $U_w$ and is isomorphic, as a variety defined over $k_0$, to a product $G_a^a(\tau) \times G_m^{b(\tau)}$, where $G_a$ and $G_m$ denote the additive and multiplicative groups of $k$, respectively, and $a(\tau)$ and $b(\tau)$ are nonnegative integers defined by the subexpression $\tau$.

In case $w' = w_0$, the element of $W$ of maximal length, a similar problem was solved by Deodhar [3]. In the general case, the result gives a geometric interpretation of a result of Kawanaka [4]. He considered the situation where $k_0$ is a finite field, and proved that the structure constant $[b_w b_{w'} : b_{w''}]$ for standard basis elements of the Iwahori algebra (or Hecke algebra) $H(G(k_0), B(k_0))$ is given by

$$[b_w b_{w'} : b_{w''}] = \sum_{\tau} q^{a(\tau)}(q - 1)^{b(\tau)}.$$

A corollary of our result gives a description of each set $U_\tau(k_0)$ in terms of root subgroups of $G(k_0)$, which implies Kawanaka's formula, and will be applied in a subsequent paper to describe the multiplication in the Hecke algebra of the Gelfand-Graev representation of $G(k_0)$, in case $k_0$ is a finite field, and the center of $G$ is

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connected. Another interesting problem would be to describe the closure patterns of the subsets \( \{U_r\} \).

The author thanks N. Kawanaka and V. Deodhar for discussions, at various times, on these ideas. The idea of seeking such a decomposition of the sets \( U(w, w', w'') \) comes from the author's attempt to understand the structure of the Hecke algebra of the Gelfand-Graev representation of \( G(k_0) \) for a finite field \( k_0 \) (see Yokonuma [6], Steinberg [5], and Chang [2]).

1. Statements of results. Besides the notation given in the Introduction, we require some other basic facts (for proofs, see Borel and Tits [1]). Let \( \Phi \) denote the root system of \( G \) with respect to \( T \), \( \{U_\alpha : \alpha \in \Phi\} \) the root subgroups, and \( \Phi^\pm \) the sets of positive and negative roots defined by the Borel subgroups \( B \) and \( B^- \). \( \Delta = \{\alpha_i : 1 \leq i \leq n\} \) denotes the set of simple roots in \( \Phi^+ \), corresponding to the set of fundamental reflections \( S = \{s_i : 1 \leq i \leq n\} \). For each \( x \in W \), let \( \psi_x = \Phi^+ \cap x\Phi^- \); then \( |\psi_x| = l(x) \), where \( l(x) \) is the length of \( x \) in the Coxeter system \( (W, S) \). We have

\[
U_x = \prod_{\alpha \in \psi_x} U_\alpha
\]

for an arbitrary ordering of the roots in \( \psi_x \), and, in any order, the map of the direct product of the \( \{U_\alpha, \alpha \in \psi_x\} \) to \( U_x \) is an isomorphism of \( k_0 \)-varieties. Moreover,

\[
U = U_x U_{xw_0} = U_{xw_0} U_x,
\]

and the map from the direct product \( U_x \times U_{xw_0} \) to \( U \) in an isomorphism of \( k_0 \)-varieties.

We have

\[
G = \bigcup_{x \in W} U_x \hat{x}B
\]

for a fixed cross section \( \{\hat{x} : x \in W\} \) in \( N_G(T)(k_0) \), and for each \( x \) there exists an isomorphism of \( k_0 \)-varieties \( U_x \times B \cong U_{x\hat{x}} B \) given by the product map. A decomposition similar to (1.3) holds for the group of rational points \( G(k_0) \).

Now let \( (w, w', w'') \) denote an arbitrary triple of elements of \( W \), which will remain fixed, and let \( s_l \cdots s_1 \) be a fixed reduced expression of \( w \) with factors \( s_i \in S \). We define

\[
U(w, w', w'') = \{u \in U_w : uwB \cap w''U_{(w')}^{-1}(w')^{-1} \neq \emptyset\}.
\]

As in Deodhar [3], a subexpression \( r \) of the reduced expression \( s_l \cdots s_1 \) is a sequence \( r = (r_1, \ldots, r_l, r_0) \) of elements of \( W \) with the property that \( r_i r_{i-1}^{-1} \in \{1, s_i\} \) for \( 1 \leq i \leq l \) and \( r_0 = 1 \). Thus the set of terminal elements \( \{r_i\} \) of subexpressions of \( s_l \cdots s_1 \) coincides with the set of elements \( x \in W \) such that \( x \leq w \) in the Bruhat order (see Deodhar [3]). A subexpression \( r = (r_1, \ldots, r_l, r_0) \) is called a \( K \)-sequence relative to the triple \( (s_l \cdots s_1 = w, w' w'') \) if it satisfies the conditions (2.10)(a c) of Kawanaka [4]. Let \( J_r = \{j : r_j r_{j-1}^{-1} = s_j\} \cup \{0\} \). Then \( r \) is such a \( K \)-sequence if and only if

\[
(1.5) \quad \tau_l w' = w'', \quad l(s_p r_j w') < l(r_j w')
\]
for each \( j \in J_r \) and \( p \) in the interval between \( j \) and the next element in \( J_r \) (or simply all \( p > j \) if \( j \) is the maximal element of \( J_r \)). For each \( K \)-sequence \( r \), set
\[
J_r^- = \{ j \in J_r : l(s_j r_j, w') < l(r_j, w') \},
\]
where \( j' \in J_r \) is the predecessor of \( j \), and define
\[
a(r) = |J_r^-|, \quad b(r) = l - |J_r| + 1 = \text{card} \{ j > 0 : r_j r_{j-1} = 1 \}.
\]

The main results can be stated as follows.

(1.6) **Theorem.** Let \((w, w', w'')\) be a triple of elements of \( W \), \( s_1 \cdots s_1 \) a fixed reduced expression of \( w \), and \( K = K(s_1 \cdots s_1, w', w'') \) the set of \( K \)-sequences relative to \((s_1 \cdots s_1, w', w'')\). Then \( U(w, w', w'') \) is independent of the choice of the cross section \( \{ \hat{x} \} \), and we have
\[
U(w, w', w'') = \bigcup_{r \in K} U_r \quad (\text{disjoint union}),
\]
where, for each \( r \), \( U_r \) is a locally closed \( k_0 \)-subvariety of \( U_w \). Moreover, there exist subsets \( \Phi_r \) and \( \Phi_r^* \) of \( \Phi^* \) of cardinalities \( a(r) \) and \( b(r) \) respectively, such that, for each \( r \), there exists an isomorphism of \( k_0 \)-varieties
\[
U_r \cong \prod_{\alpha \in \Phi_r} U_\alpha \times \prod_{\beta \in \Phi_r^*} U_\beta^* \quad (\text{where } U_\beta^* \text{ denotes the open set } U_\beta - \{ 1 \} \text{ for } \beta \in \Phi).
\]
In the statement of the next result, \( U(w, w', w'')(k_0) \) and \( U_r(k_0) \) denote the sets of \( k_0 \)-rational points on \( U(w, w', w'') \) and \( U_r \), respectively, for \( r \in K \).

(1.7) **Corollary.** Keep the notation of Theorem 1.6. Then
\[
U(w, w', w'')(k_0) = \bigcup_{r \in K} U_r(k_0) \quad (\text{disjoint union}),
\]
where for each \( r \), there exists a bijection
\[
U_r(k_0) \cong \prod_{\alpha \in \Phi_r} U_\alpha(k_0) \times \prod_{\beta \in \Phi_r^*} U_\beta^*(k_0).
\]

2. **Proofs.** For each simple root \( \alpha_i \in \Delta \), let \( G_i \) denote the \( k_0 \)-subgroup of semisimple rank one generated by \( \{ T, U_{\pm \alpha_i} \} \). We require the following structure equations concerning the multiplication in \( G_i \):

(2.1) **Lemma.** For each \( i, 1 \leq i \leq n \), there exist automorphisms of \( k_0 \)-varieties \( f_i, g_i : U_{\alpha_i}^* \to U_{\alpha_i}^* \) such that, for each \( u \in U_{\alpha_i}^* \),
\[
\hat{s}_i u \hat{s}_i = f_i(u) t_i(u) \hat{s}_i g_i(u),
\]
where \( t_i(u) \in T \).

The proof can readily be adapted from [3, Lemma 2.1].

We now begin the proof of Theorem 1.6, using induction on \( l(w) \). It is clear, in all cases, that \( U(w, w', w'') \) is independent of the choice of the cross section \( \{ \hat{x} \} \). In case \( l(w) = 1 \), with \( w = s_i \in S \), we have
\[
U(s_i, w', w'') = \begin{cases} 
1 & \text{if } s_i w' = w'', \ l(s_i w') > l(w'), \\
U_{\alpha_i} & \text{if } s_i w' = w'', \ l(s_i w') < l(w'), \\
U_{\alpha_i}^* & \text{if } w' = w'', \ l(s_i w') < l(w'),
\end{cases}
\]
and is empty in all other cases. Using this information, Theorem 1.6 is easily verified in this case.

For the rest of the proof, \( s_1 \cdots s_1 \) denotes a fixed reduced expression for \( w \), with \( l > 1 \). We assume the result holds for all triples \((s_{l-1} \cdots s_1, x, y)\), with \( x, y \in W \), and note that \( U(w, w', w'') \) is contained in the \( k_0 \)-subgroup \( U_w \), which can be factored

\[
(2.2) \quad U_w = U_{\alpha_1} \hat{s}_i U_{s_1 w}
\]

with uniqueness, so that there exists an isomorphism of \( k_0 \)-varieties \( U_w \cong U_{\alpha_1} \times U_{s_1 w} \).

For a given triple \((s_l \cdots s_1, w', w'')\), we consider two cases.

**Case 1.** \( l(s_l w'') < l(w'') \). In this case, by [4, proof of 2.14], the set of \( K \)-sequences \( K = K(s_l \cdots s_1, w', w'') \) is partitioned, \( K = K_1 \cup K_2 \), where \( K_1 \) consists of all sequences \( \sigma = (\sigma_l, \ldots, \sigma_1, \sigma_0) \) with \( \sigma' = (\sigma_{l-1}, \ldots, \sigma_1, \sigma_0) \in K(s_{l-1} \cdots s_1, w', s_l w'') \) and \( \sigma_l \sigma_{l-1} = s_l \), while \( K_2 \) consists of all sequences \( \tau = (\tau_l, \ldots, \tau_1, \tau_0) \in K(s_{l-1} \cdots s_1, w', w'') \) and \( \tau_l \tau_{l-1} = 1 \).

For each sequence \( \sigma \in K_1 \), we may assume the existence of \((U_{\sigma'}, \Phi_{\sigma'}, \Phi^*_{\sigma'})\) by the induction hypothesis, where \( \sigma' \in K(s_{l-1} \cdots s_1, w', s_l w'') \) is the part of \( \sigma \) defined above. We then define

\[
U_{\sigma} = \hat{s}_l^{-1} U_{\sigma'} \hat{s}_l, \quad \Phi_{\sigma} = s_l \Phi_{\sigma'}, \quad \Phi^*_{\sigma} = s_l \Phi^*_{\sigma'},
\]

so, in particular, \( U_{\sigma} \) is a locally closed \( k_0 \)-subvariety of \( U_w \). We shall prove that the union \( X_1 \) of all subsets \( \{U_{\sigma} : \sigma \in K_1\} \) is the part of \( U(w, w', w'') \) contained in \( s_l U_{s_1 w} \) (see (2.2)). This follows since, as is easily checked, an element \( u \in U_{s_1 w} \) belongs to \( U(s_l w, w', s_l w'') \) if and only if \( \hat{s}_l^{-1} u \hat{s}_l \in U(w, w', w'') \). The part of the induction hypothesis giving the structure of each subset \( U_{\sigma} \) in terms of the root subgroup \( \{U_{\alpha} : \alpha \in \Phi_{\sigma'}\} \) and \( \{U_{\beta} : \beta \in \Phi^*_{\sigma'}\} \) implies the corresponding result concerning the structure of \( U_{\sigma} \) in terms of the root subgroups from roots in \( \Phi_{\sigma} \) and \( \Phi^*_{\sigma} \). This is clear since \( U_{\sigma} \) is obtained from \( U_{\sigma'} \), and the root subgroups \( \{U_{\alpha} : \alpha \in \Phi_{\sigma'}\} \) and \( \{U_{\beta} : \beta \in \Phi^*_{\sigma'}\} \) are all obtained from those giving the structure of \( U_{\sigma'} \), by applying the inner automorphism by \( \hat{s}_l \).

We now prove that the remaining part of \( U(w, w', w'') \), \( X_2 = U(w, w', w'') - X_1 \), is the union of subsets associated with sequences \( \tau \in K_2 \). We first show that for an element \( u_w \in U_w \) of the form \( u_w = u s_l u_1 \hat{s}_l^{-1} \), with \( u \in U_{\alpha_1}, u_1 \in U_{s_1 w} \) as in (2.2), we have

\[
(2.3) \quad u_w \in U(w, w', w'') \quad \text{if and only if} \quad \pi(g_l(u)u_1) \in U(s_l w, w', w'')
\]

where \( \pi : U \to U_{s_1 w} \) is the morphism of \( k_0 \)-varieties defined by the factorization \( U = U_{s_1 w} U_{s_1 w_2} \) given by (1.2), and \( g_l : U_{\alpha_1} \to U_{\alpha_1}^* \) is the automorphism of \( k_0 \)-varieties defined by Lemma 2.1. To begin with, we have \( u_w \in U(w, w', w'') \) if and only if

\[
\hat{s}_l^{-1} s_l u s_l u_1 \hat{s}_l^{-1} w B \cap w'' U_{(w')^{-1}} (w')^{-1} \neq \emptyset.
\]

Using Lemma 2.1, this is equivalent to the statement

\[
(2.4) \quad \hat{s}_l^{-1} f_l(u) s_l t g(u) u_1 \hat{s}_l^{-1} w B \cap w'' U_{(w')^{-1}} (w')^{-1} \neq \emptyset
\]

with \( t = \hat{s}_l^{-1} t_l(u) \hat{s}_l \in T \) and \( \hat{s}_l^{-1} f_l(u) \hat{s}_l \in U_{-\alpha_i} \).

In case \( w'(w'')^{-1} s_l \alpha_i \in \Phi^- \), we see that (2.4) holds if and only if

\[
(2.5) \quad g_l(u)u_1(s_l w)' B \cap w'' U_{(w')^{-1}} (w')^{-1} \neq \emptyset
\]
since \( w'^{-1} U_{-\alpha_i} w'' \leq U \) by the hypothesis of Case 1, so the left factor of (2.4) can be absorbed in \( U_{(w')^{-1}} \), while the element \( t \) can be conjugated through the right-hand side, and then absorbed by \( B \). Applying the factorization (1.2) for \( s_l w \) to the element \( g_l(u)u_1 \), we obtain (2.3) in this case.

Now let \( w'(w'')^{-1}s_l \alpha_l \in \Phi^+ \). Then (2.4) is equivalent to

\[
(2.6) \quad tg_l(u)u_1^{-1} \dot{w}B \cap \dot{w}'' u^* U_{(w')^{-1}} \{ (\dot{w}')^{-1} \} \neq \emptyset
\]

where

\[
u^* = \dot{w}'^{-1} \dot{s}_l^{-1} f_l(u)^{-1} \dot{s}_l \dot{w}'' \in U_{(w')^{-1} w_0}.
\]

Since \( U_{(w')^{-1} w_0} U_{(w')^{-1} w_0} = U_{(w')^{-1} w_0} \), and \( (w')^{-1} U_{(w')^{-1} w_0} w' \leq B \), we find that (2.6) is equivalent to (2.5), completing the proof of (2.3).

After some preliminary remarks, we shall apply (2.3) to define the subsets \( \{ U_T : t \in K_2 \} \). Since \( k_0 \) is a perfect field and \( s_l U_{s_l w} \) is a closed \( k_0 \)-subgroup of \( U_w \), its complement \( Z_w \) is an open subvariety of \( U_w \) defined over \( k_0 \). The map \( (u, u_1) \rightarrow u s_l u_1 \dot{s}_l^{-1} \) defines an isomorphism of \( k_0 \)-varieties \( U_{s_l w}^* \times U_{s_l w} \cong Z_w \). The map \( \phi : U_{s_l w}^* \times U_{s_l w} \rightarrow U_{s_l w} \) given by \( \phi(u, u_1) = \pi(uu_1) \) for \( u \in U_{s_l w}^* \) and \( u_1 \in U_{s_l w} \) is clearly a morphism of varieties over \( k_0 \). Now let \( v \in U_{s_l w} \). For each fixed element \( u \in U_{s_l w}^* \), it is easily verified using the factorization (1.2) that \( u_1 = \pi(u^{-1}v) \) is the unique element of \( U_{s_l w} \) with the property that \( \pi(uu_1) = v \). Combining these remarks, we see that the map \( u s_l u_1 \dot{s}_l^{-1} \rightarrow \pi(g_l(u)u_1) \) is a surjective morphism of \( k_0 \)-varieties from \( Z_w \rightarrow U_{s_l w}^* \).

For each \( K \)-sequence \( \tau \in K_2 \), with \( \tau = (\tau_1, \ldots, \tau_0) \) and \( \tau' = (\tau_{-1}, \ldots, \tau_0) \) a \( K \)-sequence for the triple \( (s_{l-1} \cdots s_1, w', w'') \), we define

\[
(2.7) \quad U_{\tau} = \{ u s_l u_1 \dot{s}_l^{-1} \in U_{s_l w}^* s_l U_{s_l w} : \pi(g_l(u)u_1) \in U_{\tau'} \}
\]

and

\[
\Phi_{\tau} = s_l \Phi_{\tau'}, \quad \Phi_{\tau}' = s_l \Phi_{\tau}^* \cup \{ \alpha_l \}.
\]

By (2.3) and the induction hypothesis, it follows that \( X_2 = U(w, w', w'') \setminus X_1 \) is the disjoint union of the subsets \( \{ U_T \}, \tau \in K_2 \).

Each subset \( U_T \) is clearly a locally closed \( k_0 \)-subvariety of \( Z_w \), by the last statement in the paragraph preceding (2.7) and the induction hypothesis applied to \( U_{T'} \).

Now let

\[
Y_T = \{ (u, u_1) \in U_{s_l w}^* \times U_{s_l w} : \pi(uu_1) \in U_T \}.
\]

Then \( Y_T \) is a locally closed subvariety of \( U_{s_l w}^* \times U_{s_l w} \) defined over \( k_0 \), and the map \( u s_l u_1 \dot{s}_l^{-1} \rightarrow (g_l(u), u_1) \) defines an isomorphism of \( k_0 \)-varieties \( U_T \cong Y_T \). We next observe that the map \( \xi : Y_T \rightarrow U_{s_l w}^* \times U_T \) defined by \( \xi(u, u_1) = (u, \pi(uu_1)) \) is a surjective morphism of varieties over \( k_0 \). On the other hand, the preceding discussion shows that the map \( \eta : U_{s_l w}^* \times U_T \rightarrow Y_T \) given by \( \eta(u, v) = (u, \pi(u^{-1}v)) \) is a surjective morphism of \( k_0 \)-varieties, and that \( \xi \) and \( \eta \) are inverses of each other. It follows that \( Y_T \cong U_{s_l w}^* \times U_T \) as \( k_0 \)-varieties, and hence the isomorphism \( U_T \cong Y_T \) defines an isomorphism of \( k_0 \)-varieties

\[
U_T \cong \prod_{\alpha \in \Phi_T} U_\alpha \times \prod_{\beta \in \Phi_T^*} U_\beta^*,
\]

using the definitions of \( \Phi_T \) and \( \Phi_T^* \) after (2.7). This completes the proof of the inductive step in Case 1.
Case 2. \( l(s_l w'') > l(w'') \). In this situation, each \( K \)-sequence \( \tau = (\tau_1, \ldots, \tau_0) \) relative to \((s_l \cdots s_1, w', w'')\) has the property that \( \tau_1^{-1} = s_l \), and \( \tau' = (\tau_{l-1}, \ldots, \tau_0) \) is a \( K \)-sequence relative to \((s_{l-1} \cdots s_1, w', s_l w'')\). By [4, loc.cit.], this yields a bijection between the two sets of \( K \)-sequences. We shall prove that

\[
U(w, w', w'') = U_{\alpha_l} s_l U(s_l w, w', s_l w'') s_l^{-1}.
\]

Let \( x \in U_w \), and write \( x = u s_l u_1 s_l^{-1} \), with \( u \in U_{\alpha_l}, u_1 \in U_{s_l w} \). Then \( x \in U(w, w', w'') \) if and only if

\[
u s_l u_1 s_l^{-1} \bar{w} B \cap \bar{w}'' U_{(w')}^{-1} (\bar{w}'')^{-1} \neq \emptyset.
\]

Since \( l(s_l w'') > l(w'') \), we have \( \bar{w}''^{-1} u \bar{w}'' \in U \), and belongs either to \( U_{(w')}^{-1} \) or to \( U_{(w')}^{-1} w_0 \). In either case, it is easily verified, using (1.2) if necessary, that \( x \in U(w, w', w'') \) if and only if \( u_1 \in U(s_l w, w', s_l w'') \) which proves (2.8). Using (2.8) and the induction hypothesis, it is clear how to define the subsets \( \{U_\tau\} \) for each \( K \)-sequence \( \tau \in K(s_l \cdots s_1, w', w'') \), and corresponding sets of roots \( \Phi_\tau \) and \( \Phi_\tau^* \), so that the assertions of the theorem hold. This completes the proof of Theorem 1.6.

The proof of Corollary 1.7 is obtained by following the steps of the proof of Theorem 1.6 for the group of rational points \( G(k_0) \).

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