A GENERALIZED CAPACITY AND A UNIQUENESS THEOREM
ON THE DYADIC GROUP

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ABSTRACT. In this paper we shall introduce a generalized capacity and give
a necessary and sufficient condition for a subset of the dyadic group to be a
$U$-set for a certain class of Walsh series.

Wade [2] proved that a subset of the dyadic group is a $U$-set for $T_{\alpha}^+$ if and only
if it is of $\alpha$-capacity zero. We have already partially generalized Wade's theorem
in [4].

A dyadic interval of rank $n$, $I^n_p$, is the set of all 0-1 series, $x = (t_1, t_2, \ldots)$, such
that $\sum_{k=1}^{n} t_k/2^k = p/2^n$. $I_n(x)$ is the dyadic interval of rank $n$ which contains $x$. Hence $I^0_0$ is the dyadic group.

For convenience we shall write $x = (\sum_{k=1}^{\infty} t_k/2^k)^- \text{ if } \lim_{k \to \infty} t_k = 1$ and $x = (\sum_{k=1}^{\infty} t_k/2^k)$ otherwise. We refer the reader to [3 and 4] for details of the dyadic group.

A dyadic measure $m$ is a real valued set function satisfying the additive law
$m(I^n_p) = m(I^n_{2p+1}) + m(I^n_{p+1})$ for $n = 0, 1, \ldots, p = 0, 1, \ldots, 2^n - 1$. When $\mu \equiv 
\sum_{k=0}^{\infty} \hat{\mu}(k) w_k(x)$ is an arbitrary Walsh series, set

$$m_\mu(I^n_p) = \lim_{N \to \infty} \int_{I^n_p} \sum_{k=0}^{N} \hat{\mu}(k) w_k(x) \, dx = 1/2^n \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(p/2^n)$$

for each dyadic interval $I^n_p$. It is easy to see that $m_\mu$ is a dyadic measure and

$$\hat{\mu}(k) = \int_{I^n_0} w_k(x) m_\mu(dx) \equiv \int_{0}^{1} w_k(x) m_\mu(dx) \text{ for all } k.$$

Conversely an arbitrary dyadic measure always has its Walsh Fourier expansion [3].

Let $\varepsilon = \{\varepsilon_n\}_n$ be a sequence of positive numbers such that

$$1 > \varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_n > \cdots > 0 \text{ and } \lim_{n \to \infty} \varepsilon_n = 0.$$

Since $\sum_{k=2^n+1}^{2^{n+1}-1} w_k(x) = 2^n w_{2^n}(x)$ for $x \in I^n_0$ ($n = 0, 1, \ldots$) and = 0 otherwise,

$$\alpha(x) = 1 + \sum_{n=0}^{\infty} \varepsilon_n \sum_{k=2^n}^{2^{n+1}-1} w_k(x)$$
converges uniformly on each \( I_0^0 \setminus I_n^0 \) and \( \alpha(x) \) is a positive and integrable function. It is obvious that
\[
\int_0^1 \alpha(x) \, dx = 1, \quad \alpha(x) = \text{constant on } I_n^1, \quad n = 1, 2, \ldots, \quad \alpha(x) \geq \alpha(y),
\]
if \( x < y \), \( \alpha(x) = \sum_{n=0}^{\infty} a_n \chi_{I_n}^0(x) \), where \( \chi_I \) is the characteristic function of \( I \), and \( \sum_{n=0}^{\infty} a_n/2^n = 1 \).

For each Borel set \( E \subseteq I_0^0 \), \( \mathcal{M}(E) \) denotes the set of all Walsh Fourier series of nonnegative dyadic measures concentrated with \( E \) with total variation 1. For each \( \mu \in \mathcal{M}(E) \), the potential function of \( \mu \), \( U_\mu \), and the energy of \( \mu \), \( I_\mu \), are defined as follows:
\[
U_\mu(x) = \int_0^1 \alpha(x + t) \mu(dt) = \sum_{n=0}^{\infty} a_n \mu(I_n(x)),
\]
\[
I_\mu = \int_0^1 U_\mu(x) \mu(dx) = \sum_{n=0}^{\infty} \sum_{p=0}^{2^n-1} |\mu(I_n^p)|^2
\]
\[
= 1 + \sum_{n=0}^{\infty} \varepsilon_n \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2.
\]
\( C(E) = 1/\inf_{\mu \in \mathcal{M}(E)} ||U_\mu||_{\infty} \) is called the \( \varepsilon \)-capacity of \( E \). For a closed set \( E \) and \( x \notin E \), there exists a dyadic interval \( I_N(x) \) such that \( I_N(x) \cap E = \emptyset \) and \( \mu(I_n^p) = 0 \) for \( I_n^p \subseteq I_N(x) \). Hence we have
\[
U_\mu(x) = \sum_{n=0}^{N-1} a_n \mu(I_n(x)) < \infty.
\]
If \( U_\mu(x) = \infty \), then \( x \in E \). If there exists \( x_0 \) such that \( U_\mu(x_0) = \infty \), then \( ||U_\mu||_{\infty} = \infty \).

We shall state some fundamental lemmas without proofs. The proofs of these lemmas are quite similar to those of theorems stated in the book of Kahane and Salem [1, pp. 31-37].

**LEMMA 1.** For a closed subset of the dyadic group \( E \) we have
(i) \( C(E) > 0 \) if and only if there exists a Walsh series \( \mu \in \mathcal{M}(E) \) such that \( I_\mu < \infty \) and
(ii) \( 1/C(E) = \inf_{\mu \in \mathcal{M}(E)} I_\mu \equiv I(E) \).

**LEMMA 2.** If \( I(E) < \infty \), there exists uniquely a dyadic measure \( \mu^* \), which is called the equilibrium dyadic measure of \( E \), such that \( \mu^* \in \mathcal{M}(E) \) and \( I_{\mu^*} = I(E) \).

**LEMMA 3.** If a nonnegative dyadic measure \( \nu \) satisfies \( I_\nu < \infty \), then, for a closed subset \( E \), we have
\[
m_\nu(x \in E : U_{\mu^*}(x) < I(E)) \equiv \lim_{n \to \infty} m_\nu \left( \bigcup I_n(x) : x \in E, U_{\mu^*}(x) < I(E) \right) = 0.
\]
Let $\mathcal{U}_\varepsilon^+$ be the set of all Walsh series of nonnegative dyadic measure $\mu$ satisfying
\[
\sum_{n=0}^{\infty} \varepsilon_n \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}^*(k)|^2 < \infty
\]
where $\varepsilon = \{\varepsilon_n\}_n$ satisfies (1). When $\varepsilon_n = 1/2^{n(1-\alpha)}$, $n = 0, 1, \ldots$, $0 < \alpha < 1$, $\mathcal{U}_\varepsilon^+$ coincides with $\mathcal{T}_\alpha^+$ which was introduced by Wade in [2].

A subset of the dyadic group is said to be a $U$-set for $\mathcal{U}_\varepsilon^+$, if $\mu \in \mathcal{U}_\varepsilon^+$ and
\[
\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = 0 \quad \text{except on } E
\]

implies that $\mu$ is the zero series.

**Theorem 4.** A closed set of the dyadic group $E$ is a $U$-set for $\mathcal{U}_\varepsilon^+$ if and only if the $\varepsilon$-capacity of $E$ is zero.

To prove Theorem 4 we need some lemmas.

**Lemma 5.** If $E$ is of $\varepsilon$-capacity zero, then
\[
\limsup_{N \to \infty} \inf_{\mu \in \mathfrak{M}(E^c)} \sup_{x \in E_N} |U_\mu(x)| = \infty
\]

where $E_N = \bigcup_{x \in E} I_N(x)$.

**Proof.** The conclusion follows from the definition of the $\varepsilon$-capacity.

**Lemma 6.** If $E$ is of $\varepsilon$-capacity positive, then there exists $\mu \in \mathfrak{M}(E)$ such that $U_\mu \in L^\infty$ and $U_\mu(x) \geq 1/C(E)$ a.e. on $E$.

**Proof.** The Walsh series of the equilibrium dyadic measure of $E$ satisfies the conclusion.

**Lemma 7.** If $\mu \in \mathcal{U}_\varepsilon^+$ and $I$ is a dyadic interval, then there exist a dyadic measure $m_\gamma$ such that $\gamma \in \mathcal{U}_\varepsilon^+$ and a positive number $n_0$ such that $\sum_{k=0}^{2^n-1} \hat{\gamma}(k)w_k(x) = \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x)$ on $I$ and $= 0$ otherwise for $n > n_0$.

**Proof.** The following $m_\gamma$ satisfies the conclusion: $m_\gamma(I_n^c) = m_\mu(I_n^c)$ if $I_n^c \subset I$ and $= 0$ otherwise.

The following lemma is Lemma 4 of [4].

**Lemma 8.** If $\mu \in \mathcal{U}_\varepsilon^+$ satisfies
\[
\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = 0 \quad \text{a.e. outside } F,
\]
\[
\limsup_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right| < \infty \quad \text{everywhere on } F^c \text{ except perhaps on a countable set}
\]

when $F$ is a closed subset of the dyadic group, then for each dyadic interval $I \subset F^c$ there exists an integer $N$ such that $\sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = 0$ on $I$ for $n \geq N$.

**Proof of Theorem 4.** Assume that $E$ is not of $\varepsilon$-capacity zero. Then the equilibrium dyadic measure $m_{\mu^*}$ satisfies $\|U_{\mu^*}\|_\infty < \infty$. It is obvious that $\mu^*$ is
not the zero series and satisfies

\[
\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}^*(k)w_k(x) = 0 \quad \text{except on } E,
\]

\[
\sum_{k=0}^{2^n-1} \hat{\mu}^*(k)w_k(x) = 2^n m_{\mu^*}(I_n(x)) \geq 0.
\]

Since \( I_{\mu^*} \leq ||U_{\mu^*}||_{\infty} \) it is also clear that \( I_{\mu^*} \) is finite. Hence \( E \) is an \( M \)-set for \( \mathcal{U}_\varepsilon^+ \); this contradicts the hypothesis. Then \( E \) must be of \( \varepsilon \)-capacity zero.

To prove sufficiency let \( E \) be a closed set of \( \varepsilon \)-capacity zero and let \( \mu \in \mathcal{U}_\varepsilon^+ \). By the argument of [2, p. 314] it suffices to show \( \hat{\mu}(0) = 0 \). Set \( E_n = \bigcup_{x \in E} I_n(x) \) for all \( N \). Since \( E \) is closed, we have \( E = \bigcap_{N} E_N \). For sufficiently large \( n \),

\[\sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = 0 \quad \text{except on } E_N.\]

From the hypothesis \( m_{\mu} \) is a nonnegative dyadic measure; then \( \hat{\mu}(0) \geq 0 \). From (3) we have

\[
\hat{\mu}(0) = |\hat{\mu}(0)| = \int_{E_N} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \, dx \quad \text{for } n > N.
\]

By Lemma 5 there exists \( \mu_N^* \) such that

\[
U_{\mu_N^*}(x) \geq ||U_{\mu_N^*}||_{\infty} = \inf_{\mu \in \mathcal{M}(E_N)} \inf_{\mu \in \mathcal{M}(E_N)} ||U_{\mu}||_{\infty} = W_{\varepsilon}(E_N) \quad \text{on } E_N.
\]

Therefore

\[
|\hat{\mu}(0)| \leq 1/W_{\varepsilon}(E_N)^2 \left\{ \int_{E_N} \left( \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right) U_{\mu_N^*}(x) \, dx \right\}
\]

\[
\leq 1/W_{\varepsilon}(E_N)^2 \left\{ \hat{\mu}(0) + \sum_{k=0}^{N-1} \varepsilon_k \sum_{\nu = 2^k}^{2^{k+1}-1} \hat{\mu}_N(\nu) \hat{\mu}(\nu) \right\}.
\]

By Schwarz's inequality,

\[
|\hat{\mu}(0)|^2 \leq 1/W_{\varepsilon}(E_N)^2 \left\{ |\hat{\mu}(0)|^2 + \sum_{k=0}^{\infty} \varepsilon_k \sum_{\nu = 2^k}^{2^{k+1}-1} |\hat{\mu}_N(\nu)|^2 \right\}
\]

\[
\times \left\{ 1 + \sum_{k=0}^{N-1} \varepsilon_k \sum_{\nu = 2^k}^{2^{k+1}-1} |\hat{\mu}_N(\nu)|^2 \right\}.
\]

Since

\[
I_{\mu_N^*} = 1 + \sum_{n=0}^{\infty} \varepsilon_n \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}_N(k)|^2 = ||U_{\mu_N^*}||_{\infty} = W_{\varepsilon}(E_N)
\]

and \( \mu \in \mathcal{U}_\varepsilon^+ \), we have \( |\hat{\mu}(0)|^2 \leq \text{Constant} \cdot 1/W_{\varepsilon}(E_N) \). By Lemma 5 and the hypothesis, we have \( \lim_{N \to \infty} W_{\varepsilon}(E_N) = \infty \). Consequently we get \( \hat{\mu}(0) = 0 \).
REFERENCES


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