RELATION BETWEEN RIGHT AND LEFT INVOLUTIONS
OF A HILBERT ALGEBRA

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(Communicated by John B. Conway)

ABSTRACT. Existence of a densely defined right involution in a Hilbert algebra implies existence of a left involution.

$H^*$-algebras were introduced by W. Ambrose [1] to characterize Hilbert-Schmidt operators. This notion was generalized by M. F. Smiley [7], who showed that the structure theorems are valid also for a right $H^*$-algebra (a Hilbert algebra whose involution $x \rightarrow x^r$ satisfies the condition "$(yx, z) = (y, zx^r)$" but not the condition "$(xy, z) = (y, x^rz)$"). (Hilbert algebra here is a Banach algebra with a Hilbert space norm.) The author showed in [5, Theorem 2] that a proper right $H^*$-algebra is also a left $H^*$-algebra, i.e. it also has another involution $x \rightarrow x^l$ (a left involution) which satisfied the condition "$(xy, z) = (y, x^l z)$".

In this paper we shall show that the same is true also for the case when the involution $x \rightarrow x^r$ (the right involution) is defined on a dense subset only.

DEFINITION. Let $A$ be a Hilbert algebra ($A$ is a Banach algebra with a scalar product $( , )$ such that $(x, x) = \|x\|^2$). We shall say that $A$ is a weak right $H^*$-algebra if there is a dense subset $D_r$ of $A$ with the property that for each $x \in D_r$ there is some $x^r$ such that $(yx, z) = (y, zxr)$ for all $y, z \in A$. We define weak left $H^*$-algebra in a similar fashion.

The algebra $A$ is said to be proper if each $x^r$ is unique, i.e. $A$ has a right involution $x \rightarrow x^r$, defined on a dense subset. Note that $A$ is proper if $r(A) = \{u \in A: Au = (0)\}$ consists of zero alone. Algebra $A$ in Example 2, p. 54, of [5] is an example of weak right (as well as left) $H^*$-algebra. Also it is easy to show that each weak right $H^*$-algebra is a right complemented algebra.

THEOREM. Each proper weak right $H^*$-algebra $A$ is a proper weak left $H^*$-algebra.

PROOF. Note that the involution $x \rightarrow x^r$ of $A$, defined on a dense subset $D_r$, is closable, i.e. the closure of its graph is a graph of some mapping: it is easy to show that if $x_n \in D^r$, $x_n \rightarrow 0$ and $x^r_n \rightarrow y \in A$, then $y = 0$ (see end of section 8 in §5, Chapter I of [3]).

Now replace the scalar product $( , )$ of $A$ and the multiplication $\lambda x$ of members $x$ of $A$ and complex number $\lambda$ by $[ , ]$ and $\lambda \circ x$ respectively, where these new...
products are defined as follows:

\[ [x, y] = (y, x), \quad x, y \in A, \]
\[ \lambda \circ x = \overline{\lambda x}, \quad \lambda \text{ is a complex number.} \]

Let \( A' \) denote the algebra consisting of members of \( A \) but with these new operations. Note that \( A' \) is also a weak right \( H^* \)-algebra.

Let \( T': A \to A \) be a mapping defined by \( T'x = x^r \) and let \( T \) be the closed extension of \( T' \) (we remarked above that the map \( x \to x^r \) is closable). It follows from II in section 9 of §5 in Chapter I of [3], that \( T \) has an adjoint \( T^* \) defined on a dense subset \( D_1 \) of \( A' \). We define \( x^l = T^*x \) for each \( x \in D_1 \). It follows that \( (x^l, y) = (T^*x, y) = [x, Ty] = (y^r, x) \) for all \( y \in D_r \), from which we see that \( (x^l y, z) = (x^l, zy^r) = ((zy^r)^r, x) = (yz^r, x) = (y, xz) \) for all \( y, z \in D_r \) (and each \( x \in D_1 \)).

We leave it to the reader to deduce that \( (xy, z) = (y, x^l z) \) for all \( y, z \in A \).

To conclude the paper it is appropriate to comment that, in the presence of semisimplicity of the algebra \( A \), the above Theorem follows from Theorem 2 of [6]. However, semisimplicity of a proper weak right \( H^* \)-algebra would be much more difficult to establish (if it is true at all) than in the case of a right \( H^* \)-algebra. In the latter case it follows from the fact that each ideal (whether closed or not) contains a (right) selfadjoint element (it is easy to show that a selfadjoint element is not a generalized nilpotent).

REFERENCES


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