ON THE AREA OF THE REGION WHERE
AN ENTIRE FUNCTION IS GREATER THAN ONE

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To the memory of Professor Robert L. Long

ABSTRACT. Using Carleman's inequality, we prove that if $f$ is entire and of
finite order $\rho \geq 1$, then

$$\limsup_{r \to \infty} \frac{A(r)}{r^2} \geq \frac{\pi}{2\rho},$$

where $A(r)$ is the area of the region $\{z : |f(z)| \geq 1 \text{ and } |z| \leq r\}$.

1. Introduction. In [2], Edrei and Erdös proved the following

THEOREM A. Let $f$ be an entire function and $D = \{z : |f(z)| > B\}$ ($B > 0$). If
there exists a positive number $B$ such that the area of $D$ is finite, then

$$\frac{\ln \ln M(r,f)}{\ln r} \geq 2.$$ (1.1)

In this brief note, we will establish the following

THEOREM 1. Let $f$ be an entire function of order $\rho$, $1 < \rho < \infty$, and let $A(r)$
denote the area of the region

$$D_r = \{z : |f(z)| \geq 1, |z| \leq r\}.$$ 

Then

$$\limsup_{r \to \infty} \frac{A(r)}{r^2} \geq \frac{\pi}{2\rho}.$$ 

The method, which is different from that of Edrei and Erdös, is based on Carle-
man's famous inequality which we are about to introduce.

Let $D$ be a region on the complex plane. The boundary of $D$ consists of a finite
or infinite number of analytic curves clustering nowhere in the finite complex plane.
For any $r$, $0 < r < \infty$, we denote by $D_r$ the part of $D$ lying in $|z| < r$. Let $A_k(r)$
($k = 1, 2, \ldots, n(r)$) be the arcs of $|z| = r$ contained in $D$ and $r\theta_k(r)$ be their arc
lengths. Let $E = \{r : |z| = r \text{ is contained wholly in } D\}$ and $E^c = [0, \infty) - E$. If
$r \in E^c$, we define

$$\theta(r) = \max_k \theta_k(r).$$

For the moment, we leave $\theta(r)$ undefined if $r \in E$.

We now state the following version of Carleman's inequality due to K. Arima [1, 
p. 64].

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THEOREM B. Let $f$ be an entire function and $D$ be the region where $|f(z)| > 1$. Let $\theta(r)$ be defined as before for the region $D$. Then for any $a$, $0 < a < 1$, we have

$$\ln \ln M(r, f) > \pi \int_{E_r} \frac{dr}{r \theta(r)} - c,$$

where $E_r^c = E^c \cap [1, ar]$ and the constant $c$ depends on $a$ only.

2. New proof of Theorem A. Without loss of generality, we assume $B = 1$. We first choose an $a$, $0 < a < 1$ (a will be fixed throughout §§2 and 3). Let $E_r = E \cap [1, ar]$, and define $\theta(r) = 2\pi$ for $r \in E$. Then

$$\pi \int_{E_r} \frac{dt}{t \theta(t)} = \frac{1}{2} \int_{E_r} \frac{dt}{t} \leq \frac{1}{2} \ln ar.$$

From (1.2) and (2.1), we have

$$\pi \int_{1}^{ar} \frac{dt}{t \theta(t)} < \ln \ln M(r, f) + \frac{1}{2} \ln ar + c.$$

By Schwarz’s inequality,

$$(ar - 1)^2 = \left( \int_{1}^{ar} dt \right)^2 \leq \int_{1}^{ar} \theta(t) dt \cdot \int_{1}^{ar} \frac{dt}{t \theta(t)}.$$

We recall that $A(r) = \text{area of } \{z: |f(z)| \geq 1 \text{ and } |z| \leq r\}$. Clearly

$$\int_{1}^{ar} \theta(t) dt \leq A(ar).$$

From (2.2), (2.3) and (2.4), we obtain

$$(ar - 1)^2 < (A(ar)/\pi) \left( \ln \ln M(r, f) + \frac{1}{2} \ln ar + c \right).$$

If the area of $D$ is finite, (2.5) clearly implies (1.1). This completes the proof of Theorem A.

3. Proof of Theorem 1. Let

$$\mu = \liminf_{r \to \infty} \frac{1}{\ln r} \int_{E_r} \frac{dt}{t}.$$

Then $0 \leq \mu \leq 1$.

We will prove the following proposition which is slightly more general than Theorem 1.

PROPOSITION 1. Let $f$ be an entire function of order $\rho > 0$. Then

$$\limsup_{r \to \infty} \frac{A(r)}{r^2} \geq \pi \mu + \frac{\pi (1 - \mu)^2}{2 \rho}.$$

PROOF. From Schwarz’s inequality,

$$\left( \int_{E_r^c} \frac{dt}{t} \right)^2 \leq \int_{E_r^c} \frac{\theta(t)}{t} dt \cdot \int_{E_r^c} \frac{1}{\theta(t)} dt.$$

Combining (3.1) with Carleman’s inequality, we obtain

$$\int_{E_r^c} \frac{\theta(t)}{t} dt \geq \pi \left( \int_{E_r^c} \frac{dt}{t} \right)^2 / (\ln \ln M(r, f) + c).$$
We again define $\theta(r) = 2\pi$ for $r \in E$. Then

$$\int_{E_r} \frac{\theta(t)}{t} \, dt = \int_{1}^{r} \frac{\theta(t)}{t} \, dt - \int_{E_r} \frac{\theta(t)}{t} \, dt$$

$$= \int_{1}^{r} \frac{\theta(t)}{t} \, dt - 2\pi \int_{E_r} \frac{dt}{t}.$$  \hspace{1cm} (3.3)

Let $B(r) = \int_{1}^{r} t \theta(t) \, dt$. Clearly, $B(r) \leq A(r)$ for all $r \geq 1$. We therefore have

$$\int_{1}^{r} \frac{\theta(t)}{t} \, dt = \int_{1}^{r} \frac{dB(t)}{t^2} = 2 \int_{1}^{r} \frac{B(t)}{t^3} \, dt + K(r)$$

$$\leq 2 \int_{1}^{r} \frac{A(t)}{t^3} \, dt + K(r),$$  \hspace{1cm} (3.4)

where $K(r) = (B(ar)/a^2r^2 - B(1))$. We note that $B(r)/r^2 \leq \pi$ for all $r$. From (3.2), (3.3) and (3.4), we conclude that

$$K(r) \geq \frac{2}{\ln r} \int_{1}^{r} \frac{A(t)}{t^3} \, dt + \frac{2\pi}{\ln r} \int_{E_r} \frac{dt}{t} + \frac{1}{\ln r} \left( \int_{E_r} \frac{dt}{t} \right)^2 \left( \ln \ln M(r,f) + \frac{c}{\ln r} \right).$$  \hspace{1cm} (3.5)

It follows immediately from (3.5) that

$$\limsup_{r \to \infty} \frac{A(r)}{r^2} \geq \pi \mu + \frac{\pi(1 - \mu)^2}{2\rho}.$$  

This finishes the proof of Proposition 1.

It is easy to verify that, if $\rho \geq 1$.

$$\pi \mu + \frac{\pi(1 - \mu)^2}{2\rho} \geq \frac{\pi}{2\rho}. $$  \hspace{1cm} (3.6)

Theorem 1 follows from (3.6).

REMARK. From (3.5), we see that if $\rho = 0$, then $\mu = 1$. This yields

$$\limsup_{r \to \infty} \frac{A(r)}{r^2} = \pi.$$  

We also note that Proposition 1 gives $\mu + (1 - \mu)^2/2\rho \leq 1$ for $0 < \rho < 1$. This provides the following relation for $\mu$ and $\rho$:

$$1 - \frac{\rho}{2} \leq \mu \leq 1.$$  

We also point out here that the conclusion of Theorem 1 is sharp, as may be seen by considering Mittag-Leffler's function $E_{1/\rho}$.

REFERENCES


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