ON THE AREA OF THE REGION WHERE
AN ENTIRE FUNCTION IS GREATER THAN ONE

LI-CHIEN SHEN

(Communicated by Paul S. Muhly)

To the memory of Professor Robert L. Long

ABSTRACT. Using Carleman's inequality, we prove that if \( f \) is entire and of finite order \( \rho \geq 1 \), then
\[
\limsup_{r \to \infty} \frac{A(r)}{r^2} \geq \frac{\pi}{2\rho},
\]
where \( A(r) \) is the area of the region \( \{ z : |f(z)| \geq 1 \text{ and } |z| < r \} \).

1. Introduction. In [2], Edrei and Erdös proved the following

THEOREM A. Let \( f \) be an entire function and \( D = \{ z : |f(z)| > B \} \ (B > 0) \). If there exists a positive number \( B \) such that the area of \( D \) is finite, then
\[
\liminf_{r \to \infty} \frac{\ln \ln M(r,f)}{\ln r} \geq 2.
\]

In this brief note, we will establish the following

THEOREM 1. Let \( f \) be an entire function of order \( \rho \), \( 1 \leq \rho < \infty \), and let \( A(r) \) denote the area of the region
\[
D_r = \{ z : |f(z)| \geq 1, |z| \leq r \}.
\]
Then
\[
\limsup_{r \to \infty} \frac{A(r)}{r^2} \geq \frac{\pi}{2\rho}.
\]

The method, which is different from that of Edrei and Erdös, is based on Carleman's famous inequality which we are about to introduce.

Let \( D \) be a region on the complex plane. The boundary of \( D \) consists of a finite or infinite number of analytic curves clustering nowhere in the finite complex plane. For any \( r \), \( 0 < r < \infty \), we denote by \( D_r \) the part of \( D \) lying in \( |z| < r \). Let \( A_k(r) \) \((k = 1, 2, \ldots, n(r))\) be the arcs of \( |z| = r \) contained in \( D \) and \( r\theta_k(r) \) be their arc lengths. Let \( E = \{ r : |z| = r \text{ is contained wholly in } D \} \) and \( E^c = [0, \infty) - E \). If \( r \in E^c \), we define
\[
\theta(r) = \max_k \theta_k(r).
\]

For the moment, we leave \( \theta(r) \) undefined if \( r \in E \).

We now state the following version of Carleman's inequality due to K. Arima [1, p. 64].

Received by the editors March 27, 1986 and, in revised form, October 20, 1986.
1980 Mathematics Subject Classification (1985 Revision). Primary 30D30; Secondary 30D35.

©1988 American Mathematical Society
0002-9939/88 $1.00 + $.25 per page

68
THEOREM B. Let $f$ be an entire function and $D$ be the region where $|f(z)| > 1$. Let $\theta(r)$ be defined as before for the region $D$. Then for any $a, 0 < a < 1$, we have

$$\ln \ln M(r, f) > \pi \int_{E^c} \frac{dr}{r \theta(r)} - c,$$

where $E^c = E \cap [1, ar]$ and the constant $c$ depends on $a$ only.

2. New proof of Theorem A. Without loss of generality, we assume $B = 1$. We first choose an $a, 0 < a < 1$ (a will be fixed throughout §§2 and 3). Let $E_r = E \cap [1, ar]$, and define $\theta(r) = 2\pi$ for $r \in E$. Then

$$\pi \int_{E_r} \frac{dt}{t \theta(t)} = \frac{1}{2} \int_{E_r} \frac{dt}{t} \leq \frac{1}{2} \ln ar.$$

From (1.2) and (2.1), we have

$$\pi \int_{1}^{ar} \frac{dt}{t \theta(t)} < \ln \ln M(r, f) + \frac{1}{2} \ln ar + c.$$

By Schwarz’s inequality,

$$(ar - 1)^2 = \left( \int_{1}^{ar} dt \right)^2 \leq \int_{1}^{ar} t \theta(t) dt \cdot \int_{1}^{ar} \frac{dt}{t \theta(t)}.$$

We recall that $A(r) = \text{area of } \{z: |f(z)| \geq 1 \text{ and } |z| \leq r\}$. Clearly

$$\int_{1}^{ar} t \theta(t) dt \leq A(ar).$$

From (2.2), (2.3) and (2.4), we obtain

$$(ar - 1)^2 < (A(ar)/\pi) \left( \ln \ln M(r, f) + \frac{1}{2} \ln ar + c \right).$$

If the area of $D$ is finite, (2.5) clearly implies (1.1). This completes the proof of Theorem A.

3. Proof of Theorem 1. Let

$$\mu = \lim \inf_{r \to \infty} \frac{1}{\ln r} \int_{E_r} \frac{dt}{t}.$$

Then $0 \leq \mu \leq 1$.

We will prove the following proposition which is slightly more general than Theorem 1.

PROPOSITION 1. Let $f$ be an entire function of order $\rho > 0$. Then

$$\lim \sup_{r \to \infty} \frac{A(r)}{\rho^2} \geq \pi \mu + \frac{\pi(1 - \mu)^2}{2 \rho}.$$

PROOF. From Schwarz’s inequality,

$$\left( \int_{E^c} \frac{1}{t} dt \right)^2 \leq \int_{E^c} \frac{\theta(t)}{t^2} dt \cdot \int_{E^c} \frac{1}{t \theta(t)} dt.$$

Combining (3.1) with Carleman’s inequality, we obtain

$$\int_{E^c} \frac{\theta(t)}{t} dt \geq \pi \left( \int_{E^c} \frac{dt}{t} \right)^2 \left( \ln \ln M(r, f) + c \right).$$
We again define $\theta(r) = 2\pi$ for $r \in E$. Then

\[
\int_{E^c} \frac{\theta(t)}{t} \, dt = \int_{1}^{\infty} \frac{\theta(t)}{t} \, dt - \int_{E^c} \frac{\theta(t)}{t} \, dt = \int_{1}^{\infty} \frac{\theta(t)}{t} \, dt - 2\pi \int_{E^c} \frac{dt}{t}.
\]

Let $B(r) = \int_{1}^{r} \theta(t) \, dt$. Clearly, $B(r) \leq A(r)$ for all $r \geq 1$. We therefore have

\[
\int_{1}^{\infty} \frac{\theta(t)}{t} \, dt = \int_{1}^{\infty} \frac{dB(t)}{t^2} = 2 \int_{1}^{\infty} \frac{B(t)}{t^3} \, dt + K(r)
\]

where $K(r) = (B(\infty)/a - 2\pi + 2\pi)$. We note that $B(r)/r^2 < \pi$ for all $r$. From (3.2), (3.3) and (3.4), we conclude that

\[
K(r) = \frac{2}{\ln r} \int_{1}^{r} \frac{A(t)}{t^3} \, dt \geq \frac{2\pi}{\ln r} \int_{E^c} \frac{dt}{t} + \pi \left( \frac{1}{\ln r} \int_{E^c} \frac{dt}{t} \right)^2 \left( \ln \ln M(r, f) + c \right).
\]

It follows immediately from (3.5) that

\[
\limsup_{r \to \infty} \frac{A(r)}{r^2} \geq \frac{\pi \mu + \pi(1 - \mu)^2}{2\rho}.
\]

This finishes the proof of Proposition 1.

It is easy to verify that, if $\rho > 1$.

\[
\pi \mu + \frac{\pi(1 - \mu)^2}{2\rho} > \frac{\pi}{2\rho}.
\]

Theorem 1 follows from (3.6).

REMARK. From (3.5), we see that if $\rho = 0$, then $\mu = 1$. This yields

\[
\limsup_{r \to \infty} \frac{A(r)}{r^2} = \pi.
\]

We also note that Proposition 1 gives $\mu + (1 - \mu)^2/2\rho \leq 1$ for $0 < \rho < 1$. This provides the following relation for $\mu$ and $\rho$:

\[
1 - \frac{\rho}{2} \leq \mu \leq 1.
\]

We also point out here that the conclusion of Theorem 1 is sharp, as may be seen by considering Mittag-Leffler's function $E_{1/\rho}$.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611