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UNIQUE SOLVABILITY OF AN EXTENDED STIELTJES MOMENT PROBLEM

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ABSTRACT. Let $a_1, \dots, a_p$ be given real numbers ordered by size, and let $[\alpha, \beta]$ be a real interval disjoint from the set $\{a_1, \dots, a_p\}$. Let $\{c_j^{(i)} : j = 1, 2, \dots\}$, $i = 1, \dots, p$, be sequences of real numbers and c_0 be a real number. The extended Stieltjes moment problem is to find a distribution function ψ with all its points of increase in $[\alpha, \beta]$ such that

$$\int_\alpha^\beta d\psi(t) = c_0, \quad \int_\alpha^\beta \frac{d\psi(t)}{(t-a_i)^j} = c_j^{(i)}, \qquad i = 1, \dots, p, \ j = 1, 2, \dots.$$

Necessary and sufficient conditions for the existence of a unique solution of the problem are given. Orthogonal R-functions and Gaussian quadrature formulas play important roles in the proof.

1. Introduction. By a distribution function we shall mean a real-valued, bounded, nondecreasing function $\psi(t)$ with an infinite number of points of increase. By its Stieltjes transform we mean the function $\hat{\psi}(z) = \int_{-\infty}^{\infty} d\psi(t)/(t-z)$. If all the points of increase of ψ lie in an interval I, then $\hat{\psi}(z)$ is an analytic function in $C - \overline{I}$.

Let $a_1, a_2, \dots, a_p$ be distinct real numbers, ordered by size. We shall call the real interval $[\alpha, \beta]$ a *Stieltjes interval* for the point set $\{a_1, \dots, a_p\}$ if $(\alpha, \beta) \cap \{a_1, \dots, a_p\} = \varnothing$. (Intervals of the form $(-\infty, \beta]$ and $[\alpha, \infty)$ are allowed.)

Let $\{c_j^{(i)} : j = 1, 2, \dots\}$, $i = 1, \dots, p$, be given sequences of real numbers and c_0 be a real number. The *extended Stieltjes moment problem* (ESMP) is defined as follows: Find a distribution function $\psi(t)$ with all its points of increase in a given Stieltjes interval $[\alpha, \beta]$ such that

$$\int_\alpha^\beta d\psi(t) = c_0, \quad \int_\alpha^\beta \frac{d\psi(t)}{(t-a_i)^j} = c_j^{(i)}, \qquad j = 1, 2, \dots, \ i = 1, \dots, p.$$

In [**3**, **4**] we studied the *extended Hamburger moment problem* (EHMP), which consists of finding a distribution function with the same properties as above, except the requirements that the points of increase lie in a given interval. Conditions for existence of a solution of the ESMP were briefly discussed in [**5**]. In this paper we give necessary and sufficient conditions for the existence of a unique solution of the problem. Since the tools used in proving the existence of solutions also are involved in the discussion of the uniqueness question, we give a very brief sketch of the existence proof here.

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The treatment of the uniqueness problem is related to the treatment of the uniqueness problem for the strong Stieltjes moment problem, given in terms of continued fractions in [**1**, **2**]. For the classical Stieltjes moment problem, see e.g. [**6**, **7**].

2. Orthogonal R-functions. Let $\mathcal{R}$ denote the linear space consisting of all functions of the form

$$R(z) = \alpha_0 + \sum_{i=1}^{p} \sum_{j=1}^{N_i} \frac{\alpha_{ij}}{(z-a_i)^j}, \qquad \alpha_0, \alpha_{ij} \in \mathbf{C}. \tag{2.1}$$

Elements of $\mathcal{R}$ are called R-functions. We denote by $\mathcal{R}_{\mathbf{R}}$ the real space of all R-functions with real coefficients. A function R belongs to $\mathcal{R}$ iff it can be written in the form $R(z) = P(z)/Q(z)$, where Q is a polynomial with all its zeros among the points $a_1, \ldots, a_p$, and where P is a polynomial with $\deg P \leq \deg Q$. The spaces $\mathcal{R}$ and $\mathcal{R}_{\mathbf{R}}$ are closed under multiplication.

Every natural number n has a unique decomposition $n = q_n p + r_n$, $1 \leq r_n \leq p$. We write $q = q_n$, $r = r_n$. We denote by $\mathcal{R}_n$ the space of R-functions of the form

$$R(z) = \alpha_0 + \sum_{i=1}^{r} \sum_{j=1}^{q+1} \frac{\alpha_{ij}}{(z-a_i)^j} + \sum_{i=r+1}^{p} \sum_{j=1}^{q} \frac{\alpha_{ij}}{(z-a_i)^j}. \tag{2.2}$$

Let Φ denote the linear functional defined on $\mathcal{R}$ by

$$\Phi\left(\alpha_0 + \sum_{i=1}^{p} \sum_{j=1}^{N_i} \frac{\alpha_{ij}}{(z-a_i)^j}\right) = \alpha_0 c_0 + \sum_{i=1}^{p} \sum_{j=1}^{N_i} \alpha_{ij} c_j^{(i)}. \tag{2.3}$$

We shall say that Φ is *positive on an interval* $[a,b]$ if $\Phi(R) > 0$ for every $R \in \mathcal{R}$ such that $R(t) \geq 0$ and $R(t) \not\equiv 0$ for $t \in (a,b)$. It follows immediately that a necessary condition for the ESMP for a given Stieltjes interval $[\alpha, \beta]$ to have a solution is that Φ is positive on $[\alpha, \beta]$. *We shall in the following assume that this condition is satisfied.*

If the functional Φ is positive on some interval $[\alpha, \beta]$, it is positive on $(-\infty, \infty)$, and therefore gives rise to an inner product $\langle\ ,\ \rangle$ defined by $\langle R, S\rangle = \Phi(R \cdot S)$. By applying the Gram-Schmidt procedure to the sequence

$$\left\{1, \frac{1}{(z-a_1)}, \ldots, \frac{1}{(z-a_p)}, \frac{1}{(z-a_1)^2}, \ldots, \frac{1}{(z-a_p)^2}, \frac{1}{(z-a_1)^3}, \ldots\right\}$$

we obtain an orthonormal sequence $\{Q_n\}$ of R-functions. We note that $\langle Q_n, R\rangle = 0$ for every $R \in \mathcal{R}_{n-1}$. Furthermore Q_n may be written as $Q_n(z) = V_n(z)/N_n(z)$, where

$$N_n(z) = (z-a_1)^{q+1} \cdots (z-a_r)^{q+1} (z-a_{r+1})^q \cdots (z-a_p)^q, \tag{2.4}$$

and $V_n(z)$ is a polynomial of degree at most n.

3. Existence of solutions. When Φ is positive on a Stieltjes interval $[\alpha, \beta]$, the polynomial $V_n(z)$ is of degree n, and has n real simple zeros in (α, β) (cf. [**5**], see also [**3**]). This can be seen as follows: Let $t_1, \ldots, t_\lambda$ be all the zeros of odd order of $V_n(z)$ in (α, β). If $\lambda < n$, then the function

$$T_n(z) = \frac{(z-t_1)\cdots(z-t_\lambda)(z-a_r)}{N_n(z)}$$

belongs to $\mathcal{R}_{n-1}$, hence $\Phi(Q_nT_n) = 0$. On the other hand the function

$$T_n(z)Q_n(z) = \frac{(z-t_1)\cdots(z-t_\lambda)V_n(z)(z-a_r)}{N_n(z)^2}$$

has a fixed sign in (α, β), since $(z - a_r)$ does not change sign in (α, β). Thus $\Phi(Q_nT_n) \neq 0$, which is a contradiction. Consequently $\lambda = n$, and the result follows.

With the zeros $t_1^{(n)}, \ldots, t_n^{(n)}$ of $V_n(z)$ (or equivalently of $Q_n(z)$) there is associated a Gaussian quadrature formula. There exist positive weights $\lambda_{n,1}, \ldots, \lambda_{n,n}$ such that

$$\Phi(R) = \sum_{k=1}^{n} \lambda_{n,k} R(t_k^{(n)}) \tag{3.1}$$

for every R of the form (2.1), where $N_i \leq 2q + 2$ for $i < r$, $N_i \leq 2q$ for $i > r$, and $N_r \leq 2q + 1$ (see [**3**, Theorem 4.2, Proposition 4.4; **4**, Theorems 2.6, 2.7]). For given i, j we may then write

$$c_j^{(i)} = \Phi\left(\frac{1}{(z-a_i)^j}\right) = \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{(t_k^{(n)} - a_i)^j} \tag{3.2}$$

for sufficiently large n. Thus we may also write $c_j^{(i)} = \int_\alpha^\beta d\phi_n(t)/(t-a_i)^j$, where the nondecreasing, uniformly bounded functions $\phi_n(t)$ are defined by $\phi_n(t) = \sum\{\lambda_{n,k} \colon t_k^{(n)} \leq t\}$. Application of Helly's selection and convergence theorems show that the sequence $\{\phi_n(t)\}$ contains at least one subsequence converging to a distribution function ϕ, and the limit function ϕ of every such convergent subsequence has all its points of increase in (α, β) and satisfies

$$\int_\alpha^\beta d\phi(t) = c_0, \quad \int_\alpha^\beta \frac{d\phi(t)}{(t-a_i)^j} = c_j^{(i)}, \qquad j = 1, 2, \ldots, \; i = 1, \ldots, p.$$

(For more details, see [**3**].) Thus we have seen

THEOREM 1. *A necessary and sufficient condition for the ESMP on the given Stieltjes interval $[\alpha, \beta]$ to have a solution is that Φ is positive on $[\alpha, \beta]$.*

4. Uniqueness of solution. We define the R-function P_n associated with Q_n by

$$P_n(z) = \Phi_t\left(\frac{Q_n(t) - Q_n(z)}{t - z}\right). \tag{4.1}$$

(The subscript t indicates that the functional operates on its argument as a function of t.) The function $P_n(z)$ can be written in the form $P_n(z) = U_n(z)/N_n(z)$ where $U_n(z)$ is a polynomial of degree at most $n - 1$.

From the Gaussian quadrature formula it follows that we may write

$$P_n(z) = -\sum_{k=1}^{n} \lambda_{n,k} \frac{Q_n(z)}{t_k^{(n)} - z} = -Q_n(z) \int_\alpha^\beta \frac{d\phi_n(t)}{t - z}, \tag{4.2}$$

which means $P_n(z)/Q_n(z) = -\hat{\phi}_n(z)$.

If the ESMP has a unique solution $\psi(t)$, then $\{\phi_n(t)\}$ converges to this solution, since otherwise convergent subsequences would produce distinct solutions. It follows that $\{\hat{\phi}_n(z)\}$ converges to $\hat{\psi}(z)$ for all $z \in \mathbf{C} - [\alpha, \beta]$. The sequence $\{P_n(z)/Q_n(z)\}$ is convergent for the same values of z.

We shall now show that if $\{P_n(z)/Q_n(z)\}$ converges on an interval of the form (a_ρ, a_σ), where $(a_\rho, a_\sigma) \cap [\alpha, \beta] = \varnothing$, then the ESMP has a unique solution.

We note that the function

$$\begin{aligned} f(t) &= \frac{1}{(t-z)}\left[\frac{1}{N_{n-1}(t)} - \frac{1}{N_{n-1}(z)}\right] \\ &= \frac{N_{n-1}(z) - N_{n-1}(t)}{(t-z)N_{n-1}(t)N_{n-1}(z)} \end{aligned}$$

belongs to $\mathcal{R}_{n-1}$, and thus $\Phi(Q_n \cdot f) = 0$. This implies

$$\begin{aligned} &\Phi_t\left(\frac{1}{t-z}\left[\frac{Q_n(t)}{N_{n-1}(t)} - \frac{Q_n(z)}{N_{n-1}(z)}\right]\right) \\ &\quad = \Phi_t\left(\frac{1}{t-z}\left[\frac{Q_n(t) - Q_n(z)}{N_{n-1}(z)}\right]\right) + \Phi_t(Q_n(t)f(t)) \\ &\quad = \frac{1}{N_{n-1}(z)}P_n(z). \end{aligned}$$

Hence

$$P_n(z) = N_{n-1}(z)\Phi_t\left(\frac{1}{t-z}\left[\frac{Q_n(t)}{N_{n-1}(t)} - \frac{Q_n(z)}{N_{n-1}(z)}\right]\right). \tag{4.3}$$

We also note that the function

$$g(t) = \frac{V_n(t) - V_n(z)}{(t-z)N_{n-1}(t)}$$

belongs to $\mathcal{R}_{n-1}$, and thus

$$\Phi\left(Q_n(t)\frac{V_n(t) - V_n(z)}{(t-z)N_{n-1}(t)}\right) = 0. \tag{4.4}$$

Let ϕ be an arbitrary solution of the EHMP. From (4.3) we obtain

$$P_n(z) = N_{n-1}(z)\int_{-\infty}^{\infty} \frac{Q_n(t)}{(t-z)N_{n-1}(t)}\,d\phi(t) - Q_n(z)\hat{\phi}(z). \tag{4.5}$$

Similarly from (4.4) we get (noting that $N_n(t) = (t - a_r)N_{n-1}(t)$)

$$\int_{-\infty}^{\infty} \frac{Q_n(t)}{(t-z)N_{n-1}(t)}\,d\phi(t) = \frac{1}{V_n(z)}\int_{-\infty}^{\infty} \frac{Q_n(t)^2(t-a_r)}{t-z}\,d\phi(t). \tag{4.6}$$

Substitution of (4.6) into (4.5) and division by $Q_n(z)$ gives

$$\frac{P_n(z)}{Q_n(z)} = \frac{1}{(z-a_r)Q_n(z)^2}\int_{-\infty}^{\infty} \frac{(t-a_r)Q_n(t)^2}{t-z}\,d\phi(t) - \hat{\phi}(z). \tag{4.7}$$

Now assume that ϕ is a solution of the ESMP, so that we may write

$$-\hat{\phi}(z) = \frac{P_n(z)}{Q_n(z)} - \frac{1}{(z-a_r)Q_n(z)^2}\int_{\alpha}^{\beta} \frac{(t-a_r)Q_n(t)^2}{t-z}\,d\phi(t). \tag{4.8}$$

Let $z \in (a_\rho, a_\sigma)$ for some ρ, σ, where $(a_\rho, a_\sigma) \cap [\alpha, \beta] = \varnothing$. Then either $(t - a_\rho)$, $(t - a_\sigma)$, and $(t - z)$ are positive for all $t \in [\alpha, \beta]$, or $(t - a_\rho)$, $(t - a_\sigma)$, and $(t - z)$ are negative for all $t \in [\alpha, \beta]$. The numbers $(z - a_\rho)$ and $(z - a_\sigma)$ have opposite sign. Therefore $\hat{\phi}(z) + P_n(z)/Q_n(z)$ and $\hat{\phi}(z) + P_m(z)/Q_m(z)$ have opposite sign when n is of the form $n = pk + \rho$ and m is of the form $m = pk + \sigma$. It follows that if the sequences $\{P_{pk+\rho}(z)/Q_{pk+\rho}(z)\colon\ k = 0, 1, 2, \dots\}$ and $\{P_{pk+\sigma}(z)/Q_{pk+\sigma}(z)\colon\ k = 0, 1, 2, \dots\}$ are convergent and converge to the same value, then this value is $-\hat{\phi}(z)$. Thus if these two sequences converge to the same function on a subinterval of (a_ρ, a_σ), then this function is $-\hat{\phi}(z)$. From the uniqueness of the Stieltjes transform of the solution follows uniqueness of the solution itself. (Recall that the transform is known on $\mathbf{C} - [\alpha, \beta]$ when it is known on an interval.)

We may sum up the results in this section as follows:

THEOREM 2. *The following statements are equivalent.*

(A) *The ESMP has a unique solution.*

(B) *The sequence $P_n(z)/Q_n(z)$ converges on $\mathbf{C} - [\alpha, \beta]$.*

(C) *There exist an interval (a_ρ, a_σ) disjoint from $[\alpha, \beta]$ such that the sequences $\{P_{pk+\rho}(z)/Q_{pk+\rho}(z)\colon\ k = 0, 1, 2, \dots\}$ and $\{P_{pk+\sigma}(z)/Q_{pk+\sigma}(z)\colon\ k = 0, 1, 2, \dots\}$ converge to the same value (depending on z) for every z in some interval $(a, b) \subset (a_\rho, a_\sigma)$.*

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