PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 1, January 1988

HEREDITARY C^*-SUBALGEBRAS OF C^*-CROSSED PRODUCTS

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(Communicated by Paul S. Muhly)

ABSTRACT. Let (A, G, α) be a C^*-dynamical system. Assume that B is an α-invariant C^*- subalgebra of A. Then we shall give a necessary and sufficient condition for $B \times_\alpha G$ to be a C^*-subalgebra of $A \times_\alpha G$, where $B \times_\alpha G$ (resp. $A \times_\alpha G$) denotes a C^*-crossed product of B (resp. A) by a locally compact group G. Moreover, we shall show that if B is an α-invariant hereditary C^*-subalgebra of A, then $B \times_\alpha G$ is a hereditary C^*-subalgebra of $A \times_\alpha G$.

1. Introduction. Let (A, G, α) be a C^*-dynamical system, namely, a C^*-algebra A and a homomorphism α from a locally compact group G into the automorphism group of A such that $G \ni t \to \alpha_t(x)$ is continuous for each x in A. For a given (A, G, α), we can construct a new C^*-algebra, called the *C^*-crossed product of A by G* and denoted by $A \times_\alpha G$ (see [**4**] for the details). Let B be an α-invariant C^*-subalgebra of A. We very often require that $B \times_\alpha G$ is a C^*-subalgebra of $A \times_\alpha G$ in studying the C^*-crossed products or more widely objects in C^*-dynamical systems. If G is amenable, $B \times_\alpha G$ is always a C^*-subalgebra of $A \times_\alpha G$ (see [**4**, 7.7.7 and 7.7.9]). But $B \times_\alpha G$ is not necessarily a C^*-subalgebra of $A \times_\alpha G$ if G is not amenable. It is known that if B is an α-invariant ideal of A, then $B \times_\alpha G$ is a C^*-subalgebra of $A \times_\alpha G$ (see [**2**, Proposition 12]).

In §3, we shall give a necessary and sufficient condition for $B \times_\alpha G$ to be a C^*-subalgebra of $A \times_\alpha G$. Moreover, we shall show that if B is an α-invariant hereditary C^*-subalgebra of A, then $B \times_\alpha G$ is a hereditary C^*-subalgebra of $A \times_\alpha G$. Finally, we shall state an example where $B \times_\alpha G$ is not a C^*-subalgebra of $A \times_\alpha G$.

2. Preliminaries. Let (A, G, α) be a C^*-dynamical system. A *C^*-crossed product* $A \times_\alpha G$ for (A, G, α) is defined as the enveloping C^*-algebra of $L^1(A, G)$, the set of all Bochner integrable A-valued functions on G equipped with the following Banach $*$-algebra structure:

$$(xy)(t) = \int_G x(s)\alpha_s(y(s^{-1}t))\,ds,$$
$$x^*(t) = \Delta(t)^{-1}\alpha_t(x(t^{-1}))^*,$$
$$\|x\|_1 = \int_G \|x(s)\|\,ds,$$

where ds is the left Haar measure of G and $\Delta(t)$ is the associated modular function on G.

Received by the editors July 17, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L55.
Key words and phrases. C^*-crossed product, hereditary C^*-subalgebra, C^*-dynamical system.

Let $G \ni t \to u_t$ be the canonical unitary representation of G into the multiplier algebra $M(A \times_\alpha G)$ of $A \times_\alpha G$ with $\alpha_t(\cdot) = u_t \cdot u_t^*$. For each φ in $(A \times_\alpha G)^*$, there corresponds a function $\Phi\colon G \to A^*$ given by

$$\Phi(t)(x) = \varphi(xu_t)$$

for all t in G and x in A. The set of such functions is denoted by $B(A \times_\alpha G)$, and each element in $B(A \times_\alpha G)$ arising from a positive linear functional of $A \times_\alpha G$ is said to be *positive definite* with respect to α (Pedersen [**4**, 7.6.7]). Since we conversely see that

$$\varphi(y) = \int_G \Phi(t)(y(t))\,dt$$

for y in $L^1(A, G)$, the correspondence $\varphi \to \Phi$ is a bijection from $(A \times_\alpha G)^*$ onto $B(A \times_\alpha G)$ (see [**4**, 7.6.7] for the details). Now we denote by $B_+(A \times_\alpha G)$ the set of positive definite functions with respect to α in $B(A \times_\alpha G)$.

3. Results.

THEOREM 1. *Let (A, G, α) be a C^*-dynamical system, and let B be an α-invariant C^*-subalgebra of A. Then the following statements* (i), (ii) *are equivalent.*

(i) *$B \times_\alpha G$ is a C^*-subalgebra of $A \times_\alpha G$.*

(ii) *For any Φ in $B_+(B \times_\alpha G)$, there exists a positive definite function Ψ in $B_+(A \times_\alpha G)$ such that $\Psi(t)|_B = \Phi(t)$ for all t in G and $\|\Psi(e)\| = \|\Phi(e)\|$ for the identity e of G.*

PROOF. (i) $\Rightarrow$ (ii). Identifying A with its image in $M(A \times_\alpha G)$, we denote by u the canonical unitary representation of G into $M(A \times_\alpha G)$ satisfying $\alpha_t(a) = u_t a u_t^*$ for all a in A. Identifying $B \times_\alpha G$ with the image under its universal representation and denoting by λ the canonical unitary representation of G into $M(B \times_\alpha G)$ satisfying $\alpha_t(b) = \lambda_t b \lambda_t^*$ for all b in B, there exists an isomorphism ρ from $B \times_\alpha G$ onto its image under the universal representation of $A \times_\alpha G$ such that $\rho(b) = b$ for all b in B and $\rho(\lambda_t) = u_t$ for all t in G (cf. [**4**, 7.6.6]).

Take a positive linear functional φ of $B \times_\alpha G$ corresponding to Φ. Then there exists a positive linear functional ψ of $A \times_\alpha G$ such that $\psi|_{\rho(B \times_\alpha G)} = \varphi \circ \rho^{-1}$ and $\|\psi\| = \|\varphi\|$. We define a positive definite function Ψ in $B_+(A \times_\alpha G)$ by

$$\Psi(t)(x) = \psi(xu_t)$$

for all t in G and x in A. For b in B, we then have

$$\Psi(t)(b) = \psi(bu_t) = \varphi(\rho^{-1}(bu_t)) = \varphi(b\lambda_t) = \Phi(t)(b).$$

For x in A and f in $L^1(G)$, put $y(t) = f(t)x$, so y in $L^1(A, G)$, which is identified with

$$y = \int_G xu_t f(t)\,dt.$$

Using the Cauchy-Schwarz inequality, we easily see that

$$|\psi(y)|^2 \le \|\psi\| \|f\|_1^2 \psi(xx^*).$$

When x and f range over an approximate identity for A and an approximate identity for $L^1(G)$ respectively, it follows from [**1**, 2.1.5 and 2.7.5] that $\|\psi\| \leq \|\psi|_A\| = \|\Psi(e)\|$. This means that $\|\psi\| = \|\Psi(e)\|$. Similarly, we see that $\|\varphi\| = \|\Phi(e)\|$. Thus we obtain $\|\Psi(e)\| = \|\Phi(e)\|$.

(ii) $\Rightarrow$ (i). Take any positive linear functional φ of $B \times_\alpha G$ with $\|\varphi\| \leq 1$. We denote by Φ a positive definite function in $B_+(B \times_\alpha G)$ corresponding to φ. By the assumption, we can choose a positive definite function Ψ in $B_+(A \times_\alpha G)$ satisfying $\Psi(t)|_B = \Phi(t)$ for all t in G and $\|\Psi(e)\| = \|\Phi(e)\|$. Then there corresponds a positive linear functional ψ of $A \times_\alpha G$ to Ψ. For any x in $L^1(B,G)$, we denote by $\|x\|_{B\times_\alpha G}$ (resp. $\|x\|_{A\times_\alpha G}$) the C^*-norm of x in $B \times_\alpha G$ (resp. $A \times_\alpha G$). In order to prove the statement (i), it suffices to show that $\|x\|_{B\times_\alpha G} = \|x\|_{A\times_\alpha G}$. Now we have

$$\varphi(x^*x) = \int_G \Phi(t)(x^*x(t))\,dt = \int_G \Psi(t)(x^*x(t))\,dt = \psi(x^*x).$$

Since we have

$$\|\varphi\| = \|\Phi(e)\| = \|\Psi(e)\| = \|\psi\|,$$

we see that $\|\psi\| \leq 1$. Thus we conclude that $\|x\|_{B\times_\alpha G} \leq \|x\|_{A\times_\alpha G}$. Since the reverse inequality is clear, we complete the proof. Q.E.D.

Let (B, G, α) be a C^*-dynamical system. If α_t^{**} denotes the double transpose of α_t, then the map $G \ni t \to \alpha_t^{**}$ is a homomorphism of G into the automorphism group of the enveloping von Neumann algebra B^{**} of B.

LEMMA 2. *Let (B, G, α) be a C^*-dynamical system. Take any Φ from $B_+(B \times_\alpha G)$. Then we have*

$$\sum_{ij} \langle \Phi(s_i^{-1}s_j), \alpha^{**}_{s_i^{-1}}(x_i^*x_j)\rangle \geq 0$$

for finite sets $\{s_i\}$ in G and $\{x_i\}$ in B^{**}.

PROOF. For x_i in B^{**}, there exists a net $\{x_{i(k)}\}_k$ in B with $\|x_{i(k)}\| \leq \|x_i\|$ for all k such that the net $\{x_{i(k)}\}_k$ is σ-strongly* convergent to x_i (cf. [**5**, II, Lemma 2.5 and Theorem 4.8]). Then we see that $\{x_{i(k)}^*x_{j(k)}\}_k$ is σ-weakly convergent to $x_i^*x_j$. Since $\Phi(s_i^{-1}s_j)$ is an element in B^* and α_t^{**} is normal for all t in G, using [**4**, 7.6.8] we have

$$\begin{aligned} 0 &\leq \sum_{ij} \langle \Phi(s_i^{-1}s_j), \alpha_{s_i^{-1}}(x_{i(k)}^*x_{j(k)})\rangle \\ &= \sum_{ij} \langle \Phi(s_i^{-1}s_j), \alpha^{**}_{s_i^{-1}}(x_{i(k)}^*x_{j(k)})\rangle \\ &\to \sum_{ij} \langle \Phi(s_i^{-1}s_j), \alpha^{**}_{s_i^{-1}}(x_i^*x_j)\rangle. \end{aligned}$$

Consequently we obtain the desired result. Q.E.D.

LEMMA 3. *Let (A, G, α) be a C^*-dynamical system. Let B be an α-invariant hereditary C^*-subalgebra of A. Then $B \times_\alpha G$ is a C^*-subalgebra of $A \times_\alpha G$.*

PROOF. Let E be a conditional expectation from A^{**} onto B^{**}. E may be given by $E(x) = pxp$ for all x in A^{**}, where p is an α^{**}-invariant projection in A^{**}

with $B^{**} = pA^{**}p$. First we remark that

$$\alpha_t^{**}(E(x)) = E(\alpha_t(x))$$

for all x in A and t in G.

For any Φ in $B_+(B \times_\alpha G)$, we define a function $\Psi\colon G \to A^*$ by

$$\Psi(t)(x) = \langle \Phi(t), E(x)\rangle$$

for all x in A and t in G. Take finite sets $\{s_i\}$ from G and $\{x_i\}$ from A. Since E is completely positive, we have

$$E(x_i^* x_j) = \sum_k y_{i(k)}^* y_{j(k)}$$

for some $\{y_{i(k)}\}_k$ in B^{**} (cf. [**5**, IV. Lemma 3.1]). Then we have

$$\begin{aligned}
\sum_{ij} \Psi(s_i^{-1}s_j)(\alpha_{s_i^{-1}}(x_i^* x_j)) &= \sum_{ij} \langle \Phi(s_i^{-1}s_j), E(\alpha_{s_i^{-1}}(x_i^* x_j))\rangle \\
&= \sum_{ij} \langle \Phi(s_i^{-1}s_j), \alpha_{s_i^{-1}}^{**}(E(x_i^* x_j))\rangle \\
&= \sum_{ij} \langle \Phi(s_i^{-1}s_j), \sum_k \alpha_{s_i^{-1}}^{**}(y_{i(k)}^* y_{j(k)})\rangle \\
&= \sum_k \sum_{ij} \langle \Phi(s_i^{-1}s_j), \alpha_{s_i^{-1}}^{**}(y_{i(k)}^* y_{j(k)})\rangle \\
&\geq 0.
\end{aligned}$$

Here the last inequality follows from Lemma 2. Since Φ is bounded and norm continuous on G, it is easy to check the boundedness and norm continuity of Ψ on G. Therefore it follows from [**4**, 7.6.8] that Ψ is positive definite with respect to α. Since the rountine observations show that $\Psi(t)|_B = \Phi(t)$ for all t in G and $\|\Psi(e)\| = \|\Phi(e)\|$, we obtain the desired result from Theorem 1. Q.E.D.

THEOREM 4. *Let (A, G, α) be a C^*-dynamical system. Let B be an α-invariant hereditary C^*-subalgebra of A. Then $B \times_\alpha G$ is a hereditary C^*-subalgebra of $A \times_\alpha G$.*

PROOF. In order to prove our result, it suffices to show

$$(B \times_\alpha G)^{**} = p(A \times_\alpha G)^{**}p$$

for some open projection p in $(A \times_\alpha G)^{**}$.

Now we may write the universal representation of $A \times_\alpha G$ as the induced representation $(\pi \times u, H)$ via some covariant representation (π, u, H) of A (see [**4**, 7.6.4] for the notation of $(\pi \times u, H)$). Here we note that

$$\overline{(\pi \times u)(A \times_\alpha G)}^w = (A \times_\alpha G)^{**},$$

where $\overline{(\quad)}^w$ denotes the weak closure of $(\quad)$. Since B is a hereditary subalgebra A, we have $B^{**} = qA^{**}q$ for some open projection q in A^{**}. We denote by π^{**} the normal extension of π from A^{**} onto $\overline{\pi(A)}^w$ and put $p = \pi^{**}(q)$. Then we easily see that

$$\overline{\pi(B)}^w = p\overline{\pi(A)}^w p.$$

Since π^{**} is normal, p is a strong limit of a monotone increasing net of positive elements in $\pi(A)$. Hence, applying [**4**, 3.11.9 and 3.12.9], we easily see that p is an open projection in $(A \times_\alpha G)^{**}$. Since (π, u, H) is a covariant representation of A and q is α^{**}-invariant, we see that $u_t p u_t^* = p$ for all t in G. Hence, if we put

$$u_f = \int_G f(t) u_t \, dt$$

for all f in $L^1(G)$, we obtain

$$pu_f = u_f p.$$

Since $\overline{(\pi \times u)(A \times_\alpha G)}^w$ (resp. $\overline{(\pi \times u)(B \times_\alpha G)}^w$) is generated by $\{\pi(x)u_f | x \in A, f \in L^1(G)\}$ (resp. $\{\pi(x)u_f | x \in B, f \in L^1(G)\}$, the formula

$$p\pi(x)u_f p = p\pi(x)pu_f$$

for any x in A shows

$$p\overline{(\pi \times u)(A \times_\alpha G)}^w p = \overline{(\pi \times u)(B \times_\alpha G)}^w = (B \times_\alpha G)^{**}.$$

Thus we complete the proof. Q.E.D.

REMARK 5. As an alternative proof of the above theorem, it is also possible that we directly show by a few computations that $B \times_\alpha G$ is a closed linear span of $(B \times_\alpha G)(A \times_\alpha G)(B \times_\alpha G)$.

We end this paper by stating an example where $B \times_\alpha G$ is not a C^*-subalgebra of $A \times_\alpha G$.

EXAMPLE 6. Let G be a locally compact group whose enveloping group C^*-algebra $C^*(G)$ is not nuclear. Hence, it follows from [**3**, Theorem A] that there exist a C^*-algebra A and a C^*-subalgebra B of A such that the projective C^*-tensor product $B \otimes_{\max} C^*(G)$ can not be embedded in the projective C^*-tensor product $A \otimes_{\max} C^*(G)$. Consider a C^*-dynamical system (A, G, α), where α is the trivial action on G. Then $A \times_\alpha G$ and $B \times_\alpha G$ are nothing but $A \otimes_{\max} C^*(G)$ and $B \otimes_{\max} C^*(G)$, respectively.

Note also that when we consider a suitable nonamenable discrete group such as the free group on two generators, it is possible to find a nontrivial action for which the above is valid.

For the theory of C^*-tensor products of C^*-algebras, the reader is referred to [**3**, or **5**].

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